MA598CNUM ANALYTIC NUMBER THEORY, II. PROBLEMS 6

TO BE HANDED IN BY MONDAY 26TH APRIL 2021

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Assume the truth of the Riemann Hypothesis, so that $\psi(x) = x + O(x^{1/2}(\log x)^2)$.
(i) Prove that $\theta(x) = \psi(x) - x^{1/2} + O(x^{1/3})$.
(ii) Prove that $\theta(x) = x + O(x^{1/2}(\log x)^2)$.

A2.(i) Prove that
$$\int_2^x \frac{\mathrm{d}\psi(t)}{\log t} = \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + O(x^{1/3}).$$
(ii) Prove that
$$\pi(x) - \mathrm{li}(x) = \int_2^x \frac{\mathrm{d}(\psi(t)-t)}{\log t} - \frac{x^{1/2}}{\log x} + O\left(\frac{x^{1/2}}{(\log x)^2}\right).$$

B3.(i) Let $\rho = \beta + i\gamma$ be a non-trivial zero of $\zeta(s)$. Show that when x is large, one has
$$\int_2^x \frac{t^{\rho-1}}{(\log t)^2}\,\mathrm{d}t \ll \frac{x^\beta}{|\rho|(\log x)^2}.$$
(ii) Apply the explicit formula to deduce that
$$\int_2^x \frac{\mathrm{d}(\psi(t)-t)}{\log t} - \frac{\psi(x)-x}{\log x} \ll \sum_{\substack{\rho \\ |\gamma|\leqslant x}} \frac{x^\beta}{|\rho|^2(\log x)^2} + O(\log x).$$
(iii) Assuming the truth of the Riemann Hypothesis, deduce that
$$\pi(x) - \mathrm{li}(x) = \frac{\theta(x)-x}{\log x} + O\left(\frac{x^{1/2}}{(\log x)^2}\right),$$
and hence deduce that $\pi(x) = \mathrm{li}(x) + O(x^{1/2}\log x)$.

B4. Assume the truth of the Riemann Hypothesis.
(i) By a change of variable, show that when $\sigma > 1$ one has
$$-\frac{\zeta'}{\zeta}(s) = 2s\int_{1/2}^\infty \psi(2x)(2x)^{-s-1}\,\mathrm{d}x,$$
and hence deduce that
$$-\frac{2^s\zeta'(s)}{s\zeta(s)} = \int_1^\infty \psi(2x)x^{-s-1}\,\mathrm{d}x.$$
(ii) Apply the argument of the proof of Theorem 17.2 to show that, for each $\varepsilon > 0$, one has
$$\psi(2x) - 2\psi(x) = \Omega_\pm(x^{1/2-\varepsilon}).$$

C5. Assume the truth of the Riemann Hypothesis.

(i) Show that as $U \to \infty$, whenever $1 \leqslant u \leqslant U$ one has

$$\frac{\psi(e^u) - e^u}{e^{u/2}} = -\sum_{\substack{\rho \\ |\gamma| \leqslant e^U}} \frac{e^{i\gamma u}}{\rho} + O(e^{-u/3}).$$

(ii) Show that when ρ_1 and ρ_2 are two non-trivial zeros of $\zeta(s)$, then

$$\frac{1}{U}\int_1^U e^{i(\gamma_1-\gamma_2)u}\,\mathrm{d}u = \begin{cases} 1 - 1/U, & \text{when } \rho_1 = \rho_2, \\ O\left(\min\left\{1, \frac{1}{U|\gamma_1-\gamma_2|}\right\}\right), & \text{when } \rho_1 \neq \rho_2. \end{cases}$$

(iii) Prove that

$$\lim_{U\to\infty} \frac{1}{U}\int_0^U \left| \frac{\psi(e^u) - e^u}{e^{u/2}} \right|^2 \mathrm{d}u = \sum_{\rho'} \frac{m_{\rho'}^2}{|\rho'|^2},$$

where the sum is taken over the distinct zeros ρ' of $\zeta(s)$, and $m_{\rho'}$ denotes the multiplicity of the zero ρ'.