\( M A 598 \) \( C N U M \) \( A n a l y t i c \) \( N u m b e r \) \( T h e o r y, \ II. \) \( P r o b l e m s \) \( 6 - \) \( S o l u t i o n s. \)

**Q A 11** (i) One has
\[
\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p \leq x} \log p + \sum_{\frac{p}{k} \leq x} \log p + \cdots + \sum_{\frac{p}{k}^2 \leq x} \log p,
\]
where \( k = 2 \log x \)
\[
= \psi(x) + O\left(\frac{x}{\log x} \sum_{n \leq x} 1\right) = \psi(x) + O\left(\frac{x}{\log x} \right).
\]
Thus, on RH, one has
\[
\psi(x) = \psi(x) + O\left(\frac{x}{\log x} \right) = x + O\left(\frac{x}{\log x} \right).
\]
(ii) From above, we also have
\[
\psi(x) = \psi(x) + O\left(\frac{x}{\log x} \right) = x + O\left(\frac{x}{\log x} \right).
\]

Thus
\[
\psi(x) = x + O\left(\frac{x}{\log x} \right).
\]

**Q A 2** (i) From the relation
\[
\psi(x) = \psi(x) + O\left(\frac{x}{\log x} \right) + \cdots + O\left(\frac{x}{\log x} \right)
\]
\[
= \psi(x) + O\left(\frac{x}{\log x} \right) + O\left(\frac{x}{\log x} \right) + O\left(\frac{x}{\log x} \right)
\]
\[
= \psi(x) + O\left(\frac{x}{\log x} \right).
\]

we see that
\[
\int_2^x \frac{d\psi(t)}{\log t} = \int_2^x \frac{d\psi(t)}{t \log t} + \int_2^x \frac{d\psi(t)}{t \log t} + \int_2^x \frac{d\psi(t)}{t \log t} + \int_2^x \frac{d\psi(t)}{t \log t}
\]

\[
= \pi(x) + \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + O\left(\frac{x}{\log x} \right)
\]

\[
= \pi(x) + \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + O\left(\frac{x}{\log x} \right).
\]

(ii) From part (i), we have
\[
\pi(x) - li(x) = \int_2^x \frac{d\psi(t)}{\log t} - \frac{1}{2} \pi\left(\frac{x}{\log x} \right) + O\left(\frac{x}{\log x} \right).
\]

But by PNT we have
\[
\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log x} \right)
\]
\[
= \frac{x}{\log x} + O\left(\frac{x}{\log x} \right)
\]
\[
= \frac{x}{\log x} + O\left(\frac{x}{\log x} \right),
\]

Whence
\[
\pi(x) - li(x) = \int_2^x \frac{d\psi(t)}{\log t} - \frac{x}{\log x} + O\left(\frac{x}{\log x} \right).
\]
(i) By integrating by parts, one has
\[ \int \frac{\gamma}{y} \left( \frac{t^{p-1}}{(\log t)^2} \right) dt = \left[ \frac{-\gamma t^p}{p (\log t)^2} \right]_y^z + 2 \int y \frac{t^{p-1}}{p (\log t)^2} dt \]
\[ \leq \frac{y^p}{1p! (\log y)^2} + \frac{\gamma}{p! (\log y)^2} \int y \frac{t^{p-1}}{p!} dt \leq \frac{y^p}{1p! (\log y)^2} . \]

(ii) Taking \( T = y \) in the explicit formula, we see that for any \( y \geq t \),
\[ \psi(t) - t = -\sum_{1 \leq p \leq \log x} \frac{t^p}{p} + O\left( \frac{\log^2 t}{\log^3} \right) . \]

Then
\[ \int \frac{\psi(t) - t}{\log t} \leq \left[ \frac{\psi(t) - t}{\log t} \right]_2^x + \int \frac{\psi(t) - t}{\log t} dt \]
\[ = \frac{\psi(x) - x}{\log x} - \int \left( \sum_{2 \leq p \leq \log x} \frac{t^p}{p} \right) \frac{dt}{t (\log t)^2} + O\left( \int \frac{\log^2 t}{\log^3 t} dt \right) \]
\[ = \frac{\psi(x) - x}{\log x} - \sum_{2 \leq p \leq \log x} \frac{t^p}{p} \frac{dt}{t (\log t)^2} + O\left( \frac{(\log x)^2}{\log^3} \right) . \]

(iii) Assuming RH, one has
\[ \sum_{1 \leq p \leq x} \frac{x^p}{p (\log x)^2} \leq \frac{x^{1/2}}{(\log x)^2} \sum_{1 \leq p \leq x} \frac{1}{p (\log x)^2} \leq \frac{x^{1/2}}{(\log x)^2} . \]

Then by QA2 (i), QA1 (i)
\[ \int \frac{\psi(t) - t}{\log t} \leq \pi(x) - \text{li}(x) + \frac{1}{2} \pi(x^{1/2}) + O\left( x^{1/3} \right) . \]

Thus
\[ \pi(x) - \text{li}(x) = \Theta(x) - x + \frac{x^{1/2}}{\log x} - \frac{1}{2} \pi(x^{1/2}) + O\left( \frac{x^{1/2}}{\log x} \right) . \]

But
\[ \pi(x^{1/2}) = \frac{x^{1/2}}{\log x^{1/2}} + O\left( \frac{x^{1/2}}{(\log x)^2} \right) = \frac{x^{1/2}}{\log x} + O\left( \frac{x^{1/2}}{(\log x)^2} \right) \], whence
\[ \pi(x) - \text{li}(x) = \Theta(x) - x + O\left( \frac{x^{1/2}}{(\log x)^2} \right) . \]

By \( \text{AA}(ii) \), moreover, one has \( \theta (x) = x + O \left( x^{1/2} \log x \right) \), and so

\[
\pi(x) - \sigma(x) \leq x^{1/2} \log x \quad \Rightarrow \quad \pi(x) = \sigma(x) + O \left( x^{1/2} \log x \right)
\]

(\( \text{\textit{II}} \)). One has

\[
-\frac{5}{5'}(s) = 5 \int_1^\infty \psi(x) x^{-s-1} dx , \quad (e.s \text{ \textit{integration}}) \quad (s' > 1)
\]

so by substituting \( x = 2u \) we see that

\[
-\frac{5}{5'}(s) = 5 \int_{1/2}^\infty \psi(2u)(2u)^{-s-1} d(2u) = 2s \int_{1/2}^\infty \psi(2x)(2x)^{-s-1} dx
\]

and then

\[
-\frac{5}{5'}(s) = \int_1^\infty \psi(2x)x^{-s-1} dx .
\]

(iii) Suppose, if possible, that \( \psi(2x) - 2\psi(x) < \frac{1}{2} x^{1/2 - \varepsilon} \) for all \( x > \chi_0(\varepsilon) \). We have

\[
\int_1^\infty \left( x^{1/2 - \varepsilon} - \psi(2x) + 2 \psi(x) \right) x^{-s-1} dx = \frac{1}{s - \frac{1}{2} + \varepsilon} + \left( \frac{2^s}{5} - \frac{2}{5} \right) \frac{s'}{5}(s) ,
\]

and by Landau's lemma, the assumption \( \psi(2x) - 2\psi(x) < \frac{1}{2} x^{1/2 - \varepsilon} \) ensures that the left-hand side is analytic for \( \text{Re} (s) > \frac{1}{2} - \varepsilon \). But then, since

\[
\frac{2^s}{5} - \frac{2}{5} \left( (s-1) \log 2 + O(15-11^2) \right) \Rightarrow \frac{2^s}{5} \left( 5'/5'(s) \right) \text{ has no pole at } s = 1,
\]

we find that the left-hand side is analytic for \( \text{Re}(s) > \frac{1}{2} - \varepsilon \). But

\[
\frac{s'}{5}(s) \text{ has poles with real part exceeding } \frac{1}{2} - \varepsilon \quad (\text{in fact, at every } \frac{1}{2} + iy \text{ of } 5(s)) ,
\]

and so we have a contradiction. Then we have

\[
\psi(2x) - 2\psi(x) = \mathcal{U}_+ \left( x^{1/2 - \varepsilon} \right) .
\]

The argument proceeds similarly for the \( \mathcal{U}_- \) results. We suppose that

\[
\psi(2x) - 2\psi(x) > -\frac{1}{2} x^{1/2 - \varepsilon} \text{ for all } x > \chi_0(\varepsilon) ,
\]

where

\[
\int_1^\infty \left( x^{1/2 - \varepsilon} + \psi(2x) - 2\psi(x) \right) x^{-s-1} dx = \frac{1}{s - \frac{1}{2} + \varepsilon} - \left( \frac{2^s}{5} - \frac{2}{5} \right) \frac{s'}{5}(s) ,
\]

> \( \text{for } x > \chi_0(\varepsilon) \). Both LHS and RHS are analytic for \( \text{Re}(s) > \frac{1}{2} - \varepsilon \), and this again contradicts the poles of \( \frac{s'}{5}(s) \) with \( s = \frac{1}{2} + iy \text{ for some } \frac{1}{2} + iy \text{ of } 5(s) \). Thus

\[
\psi(2x) - 2\psi(x) = \mathcal{U}_- \left( x^{1/2 - \varepsilon} \right) .
\]
\( \psi(e^u) - e^u = - \sum_{\rho} \frac{(e^u)^{\rho}}{\rho} + O \left( \log^2(e^u) \right) \).

Thus, assuming RH, we have
\[
\frac{\psi(e^u) - e^u}{e^{u/2}} = - \sum_{\rho \leq e^u} \frac{(e^u)^{\frac{1}{2}+i\gamma}}{\rho e^{\frac{u}{2}}} + O \left( \frac{u^2}{e^{\frac{u}{2}}} \right)
\]
\[= - \sum_{\rho \leq e^u} \frac{e^{i\gamma u}}{\rho} + O(e^{-\frac{1}{2}u}). \tag{259} \]

(ii) When \( \rho_1 = \rho_2 \), one has
\[
\frac{1}{U} \int_{1}^{U} e^{i(y_1-y_2)u} \, du = \frac{1}{U} \int_{1}^{U} \, du = \frac{U-1}{U} = 1 - \frac{1}{U},
\]

and when \( \rho_1 \neq \rho_2 \), then instead
\[
\frac{1}{U} \int_{1}^{U} e^{i(y_1-y_2)u} \, du = \frac{e^{i(y_1-y_2)U} - e^{i(y_1-y_2)}}{i(y_1-y_2)U} \ll 1 \frac{1}{|\rho_1-\rho_2| U},
\]
or alternatively,
\[
\frac{1}{U} \int_{1}^{U} e^{i(y_1-y_2)u} \, du \ll \frac{1}{U} \int_{1}^{U} \, du \ll 1.
\]

Thus
\[
\frac{1}{U} \int_{1}^{U} e^{i(y_1-y_2)u} \, du = \begin{cases} \frac{1}{1-1/U}, & \text{when } \rho_1 = \rho_2, \\ 0 \left( \min \left\{ 1, \frac{1}{U|y_1-y_2|} \right\} \right), & \text{when } \rho_1 \neq \rho_2. \end{cases}
\]

(iii) We have
\[
\frac{1}{U} \int_{0}^{U} \left| \frac{\psi(e^u) - e^u}{e^{u/2}} \right|^2 \, du = \frac{1}{U} \int_{1}^{U} \left| \frac{\psi(e^u) - e^u}{e^{u/2}} \right|^2 \, du + O\left( \frac{1}{U} \right).
\]

By part (i), we may write
\[
\frac{\psi(e^u) - e^u}{e^{u/2}} = -f(u) + g(u),
\]

where
\[
f(u) = \sum_{\rho \leq e^u} \frac{e^{i\gamma u}}{\rho} \quad \text{and} \quad g(u) \ll e^{-\frac{1}{2}u}.
\]

We observe that in view of part (ii), one has
\[
\frac{1}{U} \int_1^U |f(u)|^2 \, du = \frac{1}{U} \sum_{\rho_1, \rho_2} \frac{1}{\rho_1 \rho_2} \int_1^U e^{i (y_1 - y_2) u} \, du \\
= (1 - \frac{1}{U}) \sum_{\rho} \frac{m_\rho^2}{\rho^2} + O \left( \sum_{\rho_1, \rho_2 | \rho_1 + \rho_2 (1 + U|y_1 - y_2|)} \right), \quad (\rho_1, \rho_2)
\]

The sum in the error term may be handled in a manner similar to that applied in the proof of Theorem 16.3. We have

\[
\sum_{\rho_1 \neq \rho_2} \frac{1}{|\rho_1 \rho_2| (1 + U|y_1 - y_2|)} \ll \frac{V}{U} \sum_{\rho_1 \neq \rho_2} \frac{1}{|\rho_1 \rho_2| 2 + i|y_1 - y_2|} \ll \frac{V}{U}.
\]

Take \( W = W(U) \) to increase slowly enough with \( U \), while taking \( W(U) \to \infty \) as \( U \to \infty \), that 

\[
\min_{1 \leq |y_i| \leq U} |y_1 - y_2| \geq 1/\sqrt{W}.
\]

Then since

\[
\sum_{\rho_1 \neq \rho_2} \frac{1}{|\rho_1 \rho_2| (1 + U|y_1 - y_2|)} \ll \sum_{\rho_1} \frac{\log T}{|\rho_1|^2} \ll (\log T) \cdot \frac{T \log T}{T^2},
\]

we find by summing over dyadic intervals that

\[
\sum_{\rho_1 \neq \rho_2} \frac{1}{|\rho_1 \rho_2| (1 + U|y_1 - y_2|)} = \sum_{\rho_1 \neq \rho_2} \frac{1}{|\rho_1 \rho_2| (1 + U|y_1 - y_2|)} \ll \frac{(\log W)^2}{W}.
\]

Hence

\[
\sum_{\rho_1 \neq \rho_2} \frac{1}{|\rho_1 \rho_2| (1 + U|y_1 - y_2|)} \ll \frac{1}{NU^2} + \frac{\log^2 W(U)}{W(U)} = o(1) \quad U \to \infty.
\]
Thus \[ \frac{1}{U} \int_{1}^{U} |f(u)|^2 \, du = \left(1 - \frac{1}{U} \right) \sum_{\rho' \in \rho} \frac{m_{\rho'}^2}{|\rho'|^{12}} + o(1). \]

Meanwhile, \[ \frac{1}{U} \int_{1}^{U} |g(u)|^2 \, du = \frac{1}{U} \int_{1}^{U} e^{-2\pi u} \, du \ll \frac{1}{U}. \]

Also, by Cauchy–Schwarz, \[ \frac{1}{U} \int_{1}^{U} |f(u)g(u)| \, du \leq \left( \frac{1}{U} \int_{1}^{U} |f(u)|^2 \, du \right)^{1/2} \left( \frac{1}{U} \int_{1}^{U} |g(u)|^2 \, du \right)^{1/2} \ll \left( \left(1 - \frac{1}{U} \right) \sum_{\rho' \in \rho} \frac{m_{\rho'}^2}{|\rho'|^{12}} + o(1) \right)^{1/2} \left( U^{-1} \right)^{1/2} \ll U^{-1/2}. \]

We thus conclude that \[ \frac{1}{U} \int_{0}^{U} \left| \frac{\psi(u^n) - e^n}{e^{u/2^n}} \right|^2 \, du = \frac{1}{U} \int_{1}^{U} \left| -f(u) + g(u) \right|^2 \, du + O\left(\frac{1}{U}\right) \]

\[ = \frac{1}{U} \int_{1}^{U} |f(u)|^2 \, du + O\left(\frac{1}{U} \int_{1}^{U} |f(u)g(u)| \, du + \frac{1}{U} \int_{1}^{U} |g(u)|^2 \, du \right) + O\left(\frac{1}{U}\right) \]

\[ = \left(1 - \frac{1}{U} \right) \sum_{\rho' \in \rho} \frac{m_{\rho'}^2}{|\rho'|^{12}} + o(1) + O\left(U^{-1/2}\right) \]

\[ \rightarrow \sum_{\rho' \in \rho} \frac{m_{\rho'}^2}{|\rho'|^{12}} \quad \text{as} \quad U \to \infty. \]

So, indeed, one has \[ \lim_{U \to \infty} \frac{1}{U} \int_{0}^{U} \left| \frac{\psi(u^n) - e^n}{e^{u/2^n}} \right|^2 \, du = \sum_{\rho' \in \rho} \frac{m_{\rho'}^2}{|\rho'|^{12}}. \]