① MA 598 CNUM Analytic Number Theory, II. Problems 6 - Solutions.

QA1 (ii) One has $\psi(x) = \sum_{n\le x} \Lambda(n) = \sum_{p\le x} \log p + \sum_{p^2\le x} (\log p) + \dots + \sum_{p^k\le x} (\log p)$, where $k = 2\log x$

$$= \theta(x) + O\Big(k(\log x) \sum_{n^2\le x} 1 \Big) = \theta(x) + O\big(x^{1/2}(\log x)^2\big).$$

Thus, on RH, one has $\theta(x) = \psi(x) + O(x^{1/2}(\log x)^2) = x + O(x^{1/2}(\log x)^2)$. □

(i) From above, we also have $\psi(x) = \theta(x) + \theta(x^{1/2}) + \dots + \theta(x^{1/k})$

$$\Rightarrow \quad \theta(x) = \psi(x) - \big(x^{\frac{1}{2}} + O(x^{\frac{1}{4}}(\log x)^2)\big) - \big(x^{1/3} + O(x^{\frac{1}{6}}(\log x)^2)\big) + O\Big(\sum_{\ell=4}^{k} x^{1/\ell} \log x\Big)$$

$$= \psi(x) - x^{1/2} + O\big(x^{1/3} + x^{1/4}(\log x)^2\big).$$

Thus $\theta(x) = \psi(x) - x^{1/2} + O(x^{1/3})$. □ //

QA2 (i) From the relation
$$\psi(x) = \theta(x) + \theta(x^{1/2}) + \dots + \theta(x^{1/k})$$
$$= \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + O(x^{\frac{1}{4}}(\log x)^2)$$
$$= \theta(x) + \theta(x^{1/2}) + O(x^{1/3}),$$

we see ~~that~~
$$\int_2^x \frac{d\psi(t)}{\log t} = \int_2^x \frac{d\,\theta(t)}{\log t} + \int_2^x \frac{d\,\theta(t^{1/2})}{2\log(t^{1/2})} + \int_2^x \frac{d\big(\psi(t) - \theta(t) - \theta(t^{1/2})\big)}{\log t}$$

$$\overset{\text{R-S integration}}{=} \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + \left[\frac{\psi(t) - \theta(t) - \theta(t^{1/2})}{\log t}\right]_2^x + \int_2^x \frac{\psi(t) - \theta(t) - \theta(t^{1/2})}{t(\log t)^2}\,dt$$

$$= \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + O(x^{1/3}) + O\Big(\int_2^x t^{-2/3}(\log t)^{-2}\,dt\Big)$$

$$= \pi(x) + \tfrac{1}{2}\pi(x^{1/2}) + O(x^{1/3}). \ □$$

(ii) From part (i), we have
$$\pi(x) - \mathrm{li}(x) = \int_2^x \frac{d(\psi(t) - t)}{\log t} - \tfrac{1}{2}\pi(x^{1/2}) + O(x^{1/3}).$$

But by [redacted] PNT we have
$$\pi(x^{1/2}) = \frac{x^{1/2}}{(\log x^{1/2})} + O\Big(\frac{x^{\frac{1}{2}}}{(\log x^{1/2})^2}\Big)$$
$$= \frac{2x^{1/2}}{\log x} + O\Big(\frac{x^{\frac{1}{2}}}{(\log x)^2}\Big),$$

Whence
$$\pi(x) - \mathrm{li}(x) = \int_2^x \frac{d(\psi(t) - t)}{\log t} - \frac{x^{1/2}}{\log x} + O\Big(\frac{x^{1/2}}{(\log x)^2}\Big). \ □ \ //$$

② QB3 (i) By integrating by parts, one has

$$\int_y^{2y} \frac{t^{\rho-1}}{(\log t)^2}\,dt = \left[\frac{t^\rho}{\rho(\log t)^2}\right]_y^{2y} + 2\int_y^{2y} \frac{t^{\rho-1}}{\rho(\log t)^3}\,dt$$

$$\ll \frac{y^\beta}{|\rho|(\log y)^2} + \frac{1}{(\log y)^3}\int_y^{2y} \frac{t^{\beta-1}}{|\rho|}\,dt \ll \frac{y^\beta}{|\rho|(\log y)^2}.$$

(ii) Taking $T=y$ in the explicit formula, we see ~~that~~ for any $y \geq t$,

$$\psi(t) - t = -\sum_{\substack{\rho \\ |\gamma|\leq y}} \frac{t^\rho}{\rho} + O(\log^2 t).$$

Then

$$\int_2^x \frac{d(\psi(t)-t)}{\log t} = \left[\frac{\psi(t)-t}{\log t}\right]_2^x + \int_2^x \frac{(\psi(t)-t)}{t(\log t)^2}\,dt$$

$$= \frac{\psi(x)-x}{\log x} - \int_2^x \Big(\sum_{\substack{\rho \\ |\gamma|\leq x}} \frac{t^\rho}{\rho}\Big)\frac{dt}{t(\log t)^2} + O\Big(\int_2^x \frac{\log^2 t}{t\log^2 t}\,dt\Big)$$

$$\Rightarrow \int_2^x \frac{d(\psi(t)-t)}{\log t} - \frac{\psi(x)-x}{\log x} = -\sum_{\substack{\rho \\ |\gamma|\leq x}} \frac{1}{\rho}\int_2^x \frac{t^{\rho-1}}{(\log t)^2}\,dt + O(\log x)$$

$$\overset{(i)}{\ll} \sum_{\substack{\rho \\ |\gamma|\leq x}} \frac{x^\beta}{|\rho|^2(\log x)^2} + O(\log x). \ \square$$

(iii) Assuming RH, one has

$$\sum_{\substack{\rho \\ |\gamma|\leq x}} \frac{x^\beta}{|\rho|^2(\log x)^2} \ll \frac{x^{\frac12}}{(\log x)^2}\underbrace{\sum_\rho \frac{1}{|\rho|^2}}_{\text{convergent, by comparison with } \sum_n \frac{\log n}{n^2}} \ll \frac{x^{1/2}}{(\log x)^2}.$$

Then by QA2 (i), QA1(i)

$$\int_2^x \frac{d(\psi(t)-t)}{\log t} = \pi(x) - li(x) + \tfrac12\pi(x^{\frac12}) + O(x^{1/3})$$

$$\| $$

$$\frac{\psi(x)-x}{\log x} + O\big(x^{1/2}/(\log x)^2\big) = \frac{\theta(x)+x^{1/2}-x}{\log x} + O\Big(\frac{x^{\frac12}}{(\log x)^2}\Big).$$

Thus

$$\pi(x) - li(x) = \frac{\theta(x)-x}{\log x} + \frac{x^{1/2}}{\log x} - \frac12\pi(x^{1/2}) + O\Big(\frac{x^{1/2}}{(\log x)^2}\Big).$$

But $\pi(x^{1/2}) = \dfrac{x^{1/2}}{\log x^{1/2}} + O\Big(\dfrac{x^{1/2}}{(\log x)^2}\Big) = \dfrac{2x^{1/2}}{\log x} + O\Big(\dfrac{x^{1/2}}{(\log x)^2}\Big)$, whence

$$\pi(x) - li(x) = \frac{\theta(x)-x}{\log x} + O\Big(\frac{x^{1/2}}{(\log x)^2}\Big). \ \square$$

③ By QA1(ii), moreover, one has $\theta(x) = x + O(x^{1/2}(\log x)^2)$, and so

$$\pi(x) - li(x) \ll x^{1/2}\log x \quad \Rightarrow \quad \pi(x) = li(x) + O(x^{1/2}\log x). \square$$

~~QB4~~ (i). One has
$$-\frac{\zeta'}{\zeta}(s) = s\int_1^\infty \psi(x)\, x^{-s-1}\,dx, \qquad (\text{R-S integration}) \; (\sigma>1)$$

so by substituting $x = 2u$ we see ~~that~~
$$-\frac{\zeta'}{\zeta}(s) = s\int_{1/2}^\infty \psi(2u)(2u)^{-s-1}\,d(2u) = 2s\int_{1/2}^\infty \psi(2x)(2x)^{-s-1}\,dx. \square$$

Then $\displaystyle\int_{1/2}^\infty \psi(2x)x^{-s-1}\,dx = -\frac{2^s\zeta'(s)}{s\zeta(s)}$. But $\psi(2x) = 0$ for $x<1$,

and thus
$$-\frac{2^s\zeta'(s)}{s\zeta(s)} = \int_1^\infty \psi(2x)\,x^{-s-1}\,dx. \square$$

(ii) Suppose, if possible, ~~that~~ $\psi(2x) - 2\psi(x) < \frac{1}{2}x^{\frac12-\varepsilon}$ for all $x > X_0(\varepsilon)$. We have
$$\int_1^\infty \big(\underbrace{x^{\frac12-\varepsilon} - \psi(2x) + 2\psi(x)}_{>0 \text{ for } x > X_0(\varepsilon)}\big)\,x^{-s-1}\,dx = \frac{1}{s-\frac12+\varepsilon} + \left(\frac{2^s}{s} - \frac{2}{s}\right)\frac{\zeta'}{\zeta}(s),$$

and by Landau's lemma, the assumption $\psi(2x) - 2\psi(x) < \frac12 x^{\frac12-\varepsilon}$ ensures ~~that~~ the left hand side is analytic for $Re(s) > \frac12 - \varepsilon$. But ~~then~~, since
$$\frac{2^s-2}{s} = \frac{2}{s}\left((s-1)\log 2 + O(|s-1|^2)\right) \Rightarrow \frac{2^s-2}{s}\left(\frac{\zeta'}{\zeta}(s)\right) \text{ has no pole at } s=1,$$

we find ~~that the~~ right hand side is analytic for $Re(s) > \frac12 - \varepsilon$. But $\zeta'/\zeta(s)$ has poles with real part exceeding $\frac12-\varepsilon$ (in fact, at every zero $\frac12+i\gamma$ of $\zeta(s)$), and so we have a contradiction. Thus we have
$$\psi(2x) - 2\psi(x) = \Omega_+(x^{\frac12-\varepsilon}). \square$$

The argument proceeds similarly for the Ω_- result. We suppose ~~that~~ $\psi(2x) - 2\psi(x) > -\frac12 x^{\frac12-\varepsilon}$ for all $x > x_0(\varepsilon)$, whence
$$\int_1^\infty \big(\underbrace{x^{\frac12-\varepsilon} + \psi(2x) - 2\psi(x)}_{>0 \text{ for } x>X_0(\varepsilon)}\big)\,x^{-s-1}\,dx = \frac{1}{s-\frac12+\varepsilon} - \left(\frac{2^s}{s} - \frac{2}{s}\right)\frac{\zeta'}{\zeta}(s).$$

Both lhs and rhs are analytic for $Re(s) > \frac12-\varepsilon$, and this again contradicts the poles of $\zeta'/\zeta(s)$ with $s = \frac12+i\gamma$ for zeros $\frac12+i\gamma$ of $\zeta(s)$. Thus
$$\psi(2x) - 2\psi(x) = \Omega_-(x^{\frac12-\varepsilon}). \square$$

④

<u>QC5</u> (i) By the explicit formula, we see that whenever $1 \le u \le U$, one has

$$\psi(e^u) - e^u = - \sum_{\substack{\rho \\ |\gamma| \le e^U}} \frac{(e^u)^\rho}{\rho} + O(\log^2(e^u)).$$

Thus, assuming RH, we have

$$\frac{\psi(e^u) - e^u}{e^{u/2}} = - \sum_{\substack{\rho \\ |\gamma| \le e^U}} \frac{(e^u)^{\frac12 + i\gamma}}{\rho\, e^{\frac12 u}} + O\left(\frac{u^2}{e^{\frac12 u}}\right)$$

$$= - \sum_{\substack{\rho \\ |\gamma| \le e^U}} \frac{e^{i\gamma u}}{\rho} + O(e^{-\frac13 u}). \quad \square$$

(ii) When $\rho_1 = \rho_2$, one has

$$\frac{1}{U} \int_1^U e^{i(\gamma_1 - \gamma_2)u}\, du = \frac{1}{U}\int_1^U du = \frac{U-1}{U} = 1 - \frac{1}{U},$$

and when $\rho_1 \ne \rho_2$, then instead

$$\frac{1}{U} \int_1^U e^{i(\gamma_1-\gamma_2)u}\, du = \frac{e^{i(\gamma_1-\gamma_2)U} - e^{i(\gamma_1-\gamma_2)}}{i(\gamma_1-\gamma_2)U} \ll \frac{1}{|\gamma_1-\gamma_2|U},$$

or alternatively,

$$\frac{1}{U} \int_1^U e^{i(\gamma_1-\gamma_2)u}\, du \ll \frac{1}{U}\int_1^U du \ll 1.$$

Thus

$$\frac{1}{U} \int_1^U e^{i(\gamma_1-\gamma_2)u}\, du = \begin{cases} 1 - 1/U, & \text{when } \rho_1 = \rho_2, \\ O\left(\min\left\{1, \frac{1}{U|\gamma_1-\gamma_2|}\right\}\right), & \text{when } \rho_1 \ne \rho_2. \end{cases} \quad \square$$

(iii) We have

$$\frac{1}{U} \int_0^U \left| \frac{\psi(e^u) - e^u}{e^{u/2}} \right|^2 du = \frac{1}{U} \int_1^U \left| \frac{\psi(e^u) - e^u}{e^{u/2}} \right|^2 du + O\left(\frac{1}{U}\right).$$

By part (i), we may write

$$\frac{\psi(e^u) - e^u}{e^{u/2}} = -f(u) + g(u),$$

where

$$f(u) = \sum_{\substack{\rho \\ |\gamma| \le e^U}} \frac{e^{i\gamma u}}{\rho} \quad \text{and} \quad g(u) \ll e^{-\frac13 u}.$$

We observe that in view of part (ii), one has

$$\frac{1}{U}\int_1^U |f(u)|^2\,du = \frac{1}{U}\sum_{\substack{\rho_1 \quad \rho_2\\ |\gamma_i|\le e^U}}\sum \frac{1}{\rho_1\bar{\rho}_2}\int_1^U e^{i(\gamma_1-\gamma_2)u}\,du$$

$$= \left(1-\frac{1}{U}\right)\sum_{\substack{\rho'\\ (\rho_1=\rho_2)}}\frac{m_{\rho'}^2}{|\rho'|^2} + O\Bigg(\sum_{\substack{\rho_1\ne\rho_2\\ |\gamma_i|\le e^U}}\frac{1}{|\rho_1\rho_2|(1+U|\gamma_1-\gamma_2|)}\Bigg).$$

The sum in the error term may be handled in a manner similar to that applied in the proof of Theorem 16.3. We have

$$\sum_{\substack{\rho_1\ne\rho_2\\ |\gamma_i|\le e^U\\ |\gamma_1-\gamma_2|\ge 1/V}}\frac{1}{|\rho_1\rho_2|(1+U|\gamma_1-\gamma_2|)} \ll \frac{V}{U}\sum_{\substack{\rho_1\ne\rho_2\\ |\gamma_i|\le e^U}}\frac{1}{|\rho_1\rho_2||2+i(\gamma_1-\gamma_2)|} \ll \frac{V}{U}.$$

(using proof of Theorem 16.3).

Take $W = W(U)$ to increase slowly enough with U, while taking $W(U)\to\infty$ as $U\to\infty$, ~~that~~ $\min_{|\gamma_1|,|\gamma_2|\le W}|\gamma_1-\gamma_2| \ge 1/\sqrt{U}$. Then since

$$\sum_{\substack{\rho_1\ne\rho_2\\ T<|\gamma_i|\le 2T\\ |\gamma_1-\gamma_2|\le 1/\sqrt{U}}}\frac{1}{|\rho_1\rho_2|(1+U|\gamma_1-\gamma_2|)} \ll \sum_{\substack{\rho_1\\ T<|\gamma_1|\le 2T}}\frac{\log T}{|\rho_1|^2} \ll (\log T)\cdot\frac{T\log T}{T^2},$$

we find by summing over dyadic intervals ~~that~~

$$\sum_{\substack{\rho_1\ne\rho_2\\ |\gamma_i|\le e^U\\ |\gamma_1-\gamma_2|\le 1/\sqrt{U}}}\frac{1}{|\rho_1\rho_2|(1+U|\gamma_1-\gamma_2|)} = \sum_{\substack{\rho_1\ne\rho_2\\ W<|\gamma_i|\le e^U\\ |\gamma_1-\gamma_2|\le 1/\sqrt{U}}}\frac{1}{|\rho_1\rho_2|(1+U|\gamma_1-\gamma_2|)}$$

$$\ll \frac{(\log W)^2}{W}.$$

Hence
$$\sum_{\substack{\rho_1\ne\rho_2\\ |\gamma_i|\le e^U}}\frac{1}{|\rho_1\rho_2|(1+U|\gamma_1-\gamma_2|)} \ll \frac{1}{\sqrt{U}} + \frac{\log^2 W(U)}{W(U)} = o(1) \text{ as } U\to\infty.$$

Thus
$$\frac{1}{U}\int_1^U |f(u)|^2\,du = \left(1-\frac{1}{U}\right)\sum_{\rho'}\frac{m_{\rho'}^2}{|\rho'|^2} + o(1).$$

Meanwhile,
$$\frac{1}{U}\int_1^U |g(u)|^2\,du = \frac{1}{U}\int_1^U e^{-\frac{2}{3}u}\,du \ll \frac{1}{U}.$$

Also, by Cauchy-Schwarz,
$$\begin{aligned}\frac{1}{U}\int_1^U |f(u)g(u)|\,du &\leq \left(\frac{1}{U}\int_1^U |f(u)|^2\,du\right)^{\frac12}\left(\frac{1}{U}\int_1^U |g(u)|^2\,du\right)^{\frac12}\\ &\ll \Bigg(\underbrace{\left(1-\frac{1}{U}\right)\sum_{\rho'}\frac{m_{\rho'}^2}{|\rho'|^2} + o(1)}_{\ll 1}\Bigg)^{\frac12}\left(U^{-1}\right)^{1/2}\\ &\ll U^{-1/2}.\end{aligned}$$

We ~~thus~~ conclude ~~that~~
$$\begin{aligned}\frac{1}{U}\int_0^U \left|\frac{\psi(e^u)-e^u}{e^{u/2}}\right|^2 du &= \frac{1}{U}\int_1^U |-f(u)+g(u)|^2\,du + O\left(\frac{1}{U}\right)\\ &= \frac{1}{U}\int_1^U |f(u)|^2\,du + O\Big(2\frac{1}{U}\int_1^U |f(u)g(u)|\,du\\ &\qquad + \frac{1}{U}\int_1^U |g(u)|^2\,du\Big)\\ &\qquad + O\left(\frac{1}{U}\right)\\ &= \left(1-\frac{1}{U}\right)\sum_{\rho'}\frac{m_{\rho'}^2}{|\rho'|^2} + o(1) + O\left(U^{-1/2}\right)\\ &\longrightarrow \sum_{\rho'}\frac{m_{\rho'}^2}{|\rho'|^2} \quad \text{as } U \to \infty.\end{aligned}$$

So, indeed, one has $\lim_{U\to\infty}\frac{1}{U}\int_0^U \left|\frac{\psi(e^u)-e^u}{e^{u/2}}\right|^2 du = \sum_{\rho'}\frac{m_{\rho'}^2}{|\rho'|^2}$. □ //