MA 59800CNUM Analytic Number Theory: a second course  
Credits: Three  
Time: 10:30 – 11:20 AM MWF

Prerequisites: A first course in analytic number theory and basic real and complex analysis.

**Description:** This course serves as a sequel to the first course in analytic number theory taught in Fall 2020. It will serve as a gateway to advanced topics interfacing with problems active in current research concerning the analytic theory of numbers. As such it explores the finer aspects of the distribution of prime numbers in arithmetic progression and consequences for such problems as the Goldbach and Twin Prime problems. Along the way, one encounters the distribution of zeros of the Riemann zeta-function and Dirichlet L-functions, and important estimates for exponential and character sums.

Following Dirichlet's proof in 1837 of the infinitude of primes in arithmetic progressions $b$ modulo $q$ (with $b$ and $q$ coprime), and the proof of the Prime Number Theorem by Hadamard and de la Vallee Poussin in 1896, researchers turned to examine finer questions concerning the distribution of prime numbers. How small is the smallest prime number congruent to $b$ modulo $q$? What can be said if one averages over the modulus $q$? How are such results connected with the zeros of Dirichlet L-functions? How close can one come to establishing the Riemann Hypothesis for such functions, and what would this imply about the distribution of primes? Do any of these results lead to interesting implications for the famous conjectures about primes? This is a second course in analytic number theory that explores important methods and estimates that lay the foundation for research in the modern theory. Students interested in broadening their knowledge of analytic methods relevant in harmonic analysis and analytic number theory will find much of this course useful, as will those preparing for research in the area ... and there are many beautiful results and theoretical developments along the way to maintain the interest of enthusiasts. The basic theory of Dirichlet series and the distribution of primes, as described in the Fall course Math 598: A First Course in Analytic Number Theory, will be assumed, though many of the topics of this second course may be studied independently of this material.

**Assessment:** Six or seven (short) problem sets will be offered through the semester, and class participants can demonstrate engagement with the course by any written and/or in-class presentations featuring a reasonable subset of these problems – three levels of difficulty: short problems testing basic skill-sets, extended problems integrating the essential methods of the course, and more challenging problems for enthusiasts with detailed hints available on request.

The course will be based on the instructor’s lecture notes, which in turn are based on: H. Davenport, Multiplicative Number Theory, 2nd ed., Springer-Verlag, GTM 74, 1980;

Basic Topics:
(i) The large sieve for exponential and character sums;
(ii) Exponential sums over primes;
(iii) The Goldbach problem: sums of two and three primes;
(iv) The Bombieri-Vinogradov theorem on primes in arithmetic progression;
(v) The Barban-Davenport-Halberstam theorem on the variance of primes in arithmetic progression;
(vi) Exponential sum estimates via van der Corput's methods;
(vii) The Burgess estimate for character sums in short intervals;
(viii) Mean and large values of Dirichlet series;
(ix) Approximate functional equations;
(x) Bounds for the Riemann zeta function and L-functions in the critical strip;
(xi) Exponential sum estimates via Vinogradov's methods;
(xii) The zero-free region for the Riemann zeta function and refined asymptotics in the prime number theorem;
(xiii) Zero density estimates;
(xiv) Primes in short intervals;
(xv) The Deuring-Heilbronn phenomenon;
(xvi) Linnik's theorem on the smallest prime in an arithmetic progression.

Advanced topics, depending on demand and available time, may include an introduction to sieves, Maynard's theorem on prime k-tuples, pair correlation and zero density of the zeros of the Riemann zeta function.