PURDUE UNIVERSITY

Department of Mathematics

REAL ANALYSIS
MA 50400 - SOLUTIONS

6th October 2021   50 minutes

This paper contains **SIX** *questions.*
All SIX answers will be used for assessment.
Calculators, textbooks, notes and cribsheets are **not** *permitted in this examination.*

Do not turn over until instructed.

1. [3+3+3+3+3=15 points] Decide which of the following statements are necessarily true, and which may be false. Mark those which are true with "T", and those which are false with "F".

a. Every Cauchy sequence in a complete metric space converges.

TRUE (complete metric spaces contain all limits of Cauchy sequences, by definition).

b. ϵ is the smallest positive real number.

FALSE (if true, $\epsilon/2$ would be smaller than ϵ, giving a contradiction).

c. No open cover of $(0, 1)$ contains a finite subcover.

FALSE (for example, the set $(0, 1)$ forms an open cover of itself).

d. Given a real sequence (s_n), the phrase "(s_n) converges to s" means: given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n > N$ then $|s_n - s| < \epsilon$.

TRUE (this is the definition).

e. Given a real sequence (s_n), the phrase "(s_n) converges to s" means: given any $\epsilon > 0$ there exists $N \in \mathbb{N}$ and some $n > N$ such that $|s_n - s| < \epsilon$.

FALSE (this is **not** the definition).

2. [4+4+4+4=16 points]

(a) Define what is meant by an *open* set E in a metric space (X, d).

E is an open set if every point in E is an interior point. (A point $p \in X$ is an interior point of E if, for some positive number r, there is a neighborhood of p with radius r contained within E).

(b) Define what is meant by a *limit point* of a set E in a metric space (X, d).

A point $p \in X$ is a limit point of E if every neighborhood of p contains a point $q \in E$ with $q \neq p$.

(c) Define what is meant by an *open cover* of a set E in a metric space X.

An open cover of E is a collection of open sets $\{G_\alpha\}_{\alpha \in A}$, with $E \subset \cup_{\alpha \in A} G_\alpha$.

(d) Define what is meant by a *compact* subset K of a metric space X.

K is compact if every open cover of K contains a finite subcover of K.

Continued...

3. [8+8+8=24 points] This question concerns the metric space $\mathbb{R}$ with the usual absolute value.

(a) Let (n_k) be any sequence of integers with $1 = n_1 < n_2 < \ldots$. Show that the collection of sets $(-n_k, -1) \cup (-1/n_k, 1/n_k)$, with $k \in \mathbb{N}$, forms an open cover of each of the sets

$$E_1 = \{x \in \mathbb{R} : x < -1\} \quad \text{and} \quad E_2 = \{0\} \cup \left\{ \frac{1}{n+1} : n \in \mathbb{N} \right\}.$$

The first observation is that since for each $k \in \mathbb{N}$, the sets $(-n_k, -1)$ and $(-1/n_k, 1/n_k)$ are both open, then their union (say G_k) is also open.

Next we check that $\cup_{k\in\mathbb{N}} G_k$ forms an open cover of E_1. If $x \in E_1$, then $x < -1$. We take n_k to be any integer larger than $-x$ (since (n_k) is monotonic increasing, there exists such an n_k), and then observe that $x \in (-n_k, -1)$, whence $x \in G_k$. Thus x is indeed contained in the union of the open sets G_k with $k \in \mathbb{N}$, so that the claimed collection forms an open cover of E_1.

Finally, we check that $\cup_{k\in\mathbb{N}} G_k$ forms an open cover of E_2. If $x \in E_2$, then either $x = 0$ or $x = 1/(n+1)$ for some $n \in \mathbb{N}$. Thus, in either case, we have $x \in [-1/2, 1/2]$, and so $x \in (-1/n_1, 1/n_1) = (-1, 1) \subset G_1$. Thus x is indeed contained in the union of the open sets G_k with $k \in \mathbb{N}$, so that the claimed collection forms an open cover of E_2.

(b) Is the set E_1 defined in part (a) compact? Justify your answer.

The simplest approach is to make use of the Heine-Borel theorem, which shows that the set E_1 is compact if and only if it is closed and bounded. But the set E_1 is not bounded (if all elements x of E_1 were bounded in absolute value by $M > 0$, then by considering $-2M \in E_1$ we obtain a contradiction). Since E_1 is not bounded, it cannot be compact.

[One can also proceed directly: suppose that $\cup_{k\in K} G_{n_k}$ is any finite subcover of the open cover in (a). Take N to be the largest of the integers n_k with $k \in K$ and consider the point $x = -2N \in E_1$. Then x is not contained in any of the sets G_{n_k} $(k \in K)$, so E_1 is not covered by the putative finite subcover. This contradiction shows that E_1 is not compact.]

(c) Is the set E_2 defined in part (a) compact? Justify your answer.

The simplest approach is again via the Heine-Borel theorem. The set E_2 is contained in $[-1/2, 1/2]$ so is bounded (every element has absolute value smaller than 1). In addition the set E_2 is closed, because it contains all of its limit points: the only limit point is $0 \in E_2$. Thus E_2 is closed and bounded, and hence compact.

[One can also proceed directly: suppose that $\cup_{a\in A} B_a$ is *any* open cover of E_2. Some one of these open sets B_{a_0} contains 0, and since 0 is an interior point of B_{a_0}, there is a neighborhood of 0 of some radius $r > 0$ contained inside B_{a_0}. Any point $1/(n+1)$ contained in E_2 with $n > 1/r$ is contained inside this neighborhood inside B_{a_0}. Then if $B_{a_1}, \ldots, B_{a_m}$ are members of the collection of open sets respectively containing $1/2, 1/3, \ldots, 1/([1/r]+1)$, then we find that E_2 is contained inside the finite subcover $\cup_{i=0}^{m} B_{a_i}$. Thus E_2 is indeed compact.]

Continued...

4. [15 points] Suppose that $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence in a metric space (X, d), and some subsequence $(p_{n_k})_{k=1}^{\infty}$ converges to a point $p \in X$. Prove that the full subsequence $(p_n)_{n=1}^{\infty}$ converges to p.

   Since $(p_{n_k})_{k=1}^{\infty}$ converges to the point $p \in X$, we see that for all $\epsilon > 0$, there exists $K = K(\epsilon) \in \mathbb{N}$ such that, whenever $k \geq K(\epsilon)$, one has $d(p_{n_k}, p) < \epsilon$.

   Likewise, since $(p_n)_{n=1}^{\infty}$ is a Cauchy sequence, for all $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that, whenever $m \geq n \geq N(\epsilon)$, one has $d(p_m, p_n) < \epsilon$.

   Thus, given $\epsilon > 0$, whenever $n \geq N(\epsilon/2)$, we find from the triangle inequality that by taking $k \geq N(\epsilon/2) + K(\epsilon/2)$ (which implies that $n_k \geq N(\epsilon/2)$) and $m = n_k$, then one has

$$d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \epsilon/2 + \epsilon/2 = \epsilon.$$

   Thus, whenever $\epsilon > 0$, there exists $N' = N'(\epsilon) \in \mathbb{N}$ such that, whenever $n > N'(\epsilon)$, one has $d(p_n, p) < \epsilon$. Thus (p_n) does indeed converge to p.

5. [15 points] Let (a_n) be a convergent sequence of real numbers with $\lim a_n = a$. Put

$$b_n = \frac{a_n}{1 + a_n^2}.$$

   Prove that the sequence (b_n) converges, and determine its limit.

   The simplest approach is to use the arithmetic of sequences. Since $\lim a_n = a$, we have $\lim a_n^2 = (\lim a_n)^2 = a^2$ and hence $\lim(1 + a_n^2) = 1 + \lim a_n^2 = 1 + a^2$. Thus (important!) since $1 + a_n^2 \neq 0$ and $1 + a^2 \neq 0$ (both are at least 1), we find that

$$\lim \frac{1}{1 + a_n^2} = \frac{1}{\lim(1 + a_n^2)} = \frac{1}{1 + a^2}.$$

   Finally, again using arithmetic of sequences,

$$\lim \frac{a_n}{1 + a_n^2} = \frac{\lim a_n}{\lim(1 + a_n^2)} = \frac{a}{1 + a^2}.$$

   Thus (b_n) does indeed converge, and has limit $a/(1 + a^2)$.

   One can also proceed directly. The convergence of (a_n) implies that, for each $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{N}$ such that, whenever $n \geq N(\epsilon)$, one has $|a_n - a| < \epsilon$. Thus, whenever $n \geq N(\epsilon)$, one has

$$|b_n - b| = \left| \frac{a_n}{1 + a_n^2} - \frac{a}{1 + a^2} \right| = \left| \frac{(a_n - a)(1 - a_n a)}{(1 + a_n^2)(1 + a^2)} \right|.$$

   Thus, since $|1 - a_n a| \leq 1 + |a|^2 + |a_n^2| \leq (1 + a_n^2)(1 + a^2)$, we see that $|b_n - b| \leq |a_n - a| < \epsilon$, and hence $\lim b_n$ exists, and is equal to $a/(1 + a^2)$.

Continued...

6. [15 points] Suppose that (b_n) is a sequence of positive real numbers having the property that
$$\liminf_{n\to\infty} \frac{b_n}{n} > 0.$$
Prove that the series
$$\sum_{n=1}^{\infty} \frac{1}{b_n^2}$$
converges.

We may suppose that there is a positive number β for which
$$\liminf_{n\to\infty} \frac{b_n}{n} = \lim_{n\to\infty} \inf \left\{ \frac{b_{n+k}}{n+k} : k \in \mathbb{N} \right\} > \beta > 0.$$
Thus, for all large enough natural numbers n, one has $b_n/n \geq \beta$, whence $1/b_n^2 \leq 1/(\beta n)^2$. Thus, by the comparison test, we find that $\sum 1/b_n^2$ converges, since $\sum 1/n^2$ converges.

End of examination.