PURDUE UNIVERSITY

Department of Mathematics

REAL ANALYSIS

MA 50400 - SOLUTIONS

17th November 2021   50 minutes

This paper contains **SIX** *questions.*
All SIX answers will be used for assessment.
Calculators, textbooks, notes and cribsheets are **not** *permitted in this examination.*

Do not turn over until instructed.

1. [3+3+3+3+3=15 points] Decide which of the following statements are necessarily true, and which may be false. Mark those which are true with “T”, and those which are false with “F”.

   a. If (a_n) is a real sequence satisfying $a_1 \geq a_2 \geq \cdots \geq a_n \geq \cdots \geq 0$, then the series $\sum(-1)^n a_n$ is convergent.

   FALSE (consider the sequence (a_n) with $a_n = 1$ $(n \in \mathbb{N})$, since $\sum(-1)^n$ is not convergent).

   b. Let $g : \mathbb{R} \to \mathbb{R}$. If there exists a real sequence (x_n) such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} g(x_n) = g(1)$, then g is continuous at 1.

   FALSE (Consider the sequence (x_n) with $x_n = 1+1/n$ $(n \in \mathbb{N})$, and the function $g : \mathbb{R} \to \mathbb{R}$ given by taking $g(x) = 0$ when $x \in \mathbb{Q}$ and $g(x) = 1$ when $x \in \mathbb{R} \setminus \mathbb{Q}$).

   c. if h is a real-valued function defined on $\mathbb{R}$ and h is continuous at a, then $\lim_{x\to a} h(x)$ exists.

   TRUE (if h is continuous, then $\lim_{x\to a} h(x) = h(a)$, so the limit exists).

   d. Let $u : \mathbb{R} \to \mathbb{R}$ be continuous at a point a. Then u is differentiable at a.

   FALSE (consider $u(x) = |x|$ for $x \in \mathbb{R}$, which is continuous but not differentiable at $x = 0$).

   e. A function $f : [a, b] \to \mathbb{R}$ which is differentiable on $[a, b]$ is necessarily Riemann integrable on $[a, b]$.

   TRUE (a function which is differentiable on $[a, b]$ is continuous on $[a, b]$ and hence Riemann integrable on $[a, b]$).

2. [5+5+5=15 points]

   (a) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Define what it means for f to be *continuous* at p.

   For every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, p) > 0$ such that, whenever $x \in E$ and $d_X(x, p) < \delta$, then $d_Y(f(x), f(p)) < \epsilon$.

   (b) Suppose that (X, d_X) and (Y, d_Y) are metric spaces, and f maps X into Y. Define what it means for f to be *uniformly continuous* on X.

   For every $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, whenever $p, q \in X$ and $d_X(p, q) < \delta$, then $d_Y(f(p), f(q)) < \epsilon$.

   (c) Let $f : [a, b] \to \mathbb{R}$. Define what is meant by the *derivative* f' of f at $x \in [a, b]$.

   Define
   $$\phi(t) = \frac{f(t) - f(x)}{t - x} \quad (t \in (a, b) \setminus \{x\}).$$
   Then $f'(x) = \lim_{t\to x} \phi(t)$, provided that this limit exists.

Continued...

3. [8+8+8+1=25 points] The function $\lfloor x \rfloor$ is defined to be the largest integer not exceeding x. For example, $\lfloor 10/3 \rfloor = 3$, $\lfloor -5/2 \rfloor = -3$ and $\lfloor 2 \rfloor = 2$. The function $g(x)$ is defined for real numbers x by

$$g(x) = \begin{cases} x^2 \lfloor 1/x \rfloor, & \text{when } x \neq 0, \\ 0, & \text{when } x = 0. \end{cases}$$

(a) Show that g is continuous at $x = 0$.

For each $x \in \mathbb{R} \setminus \{0\}$, one has $1/x - 1 \leq \lfloor 1/x \rfloor \leq 1/x$. Thus

$$x - x^2 = x^2(1/x - 1) \leq g(x) \leq x^2(1/x) = x \quad (x \neq 0),$$

whence

$$0 = \lim_{x \to 0}(x - x^2) \leq \lim_{x \to 0} g(x) \leq \lim_{x \to 0} x = 0.$$

Thus we have $\lim_{x \to 0} g(x) = 0 = g(0)$, which shows that g is continuous at $x = 0$.

(b) Show that g is differentiable at $x = 0$, and find $g'(0)$.

Consider

$$h(t) = \frac{g(t) - g(0)}{t - 0} = t \lfloor 1/t \rfloor \quad (t \neq 0).$$

Then, provided that the limit exists, we have

$$g'(0) = \lim_{t \to 0} h(t) = \lim_{t \to 0} t \lfloor 1/t \rfloor.$$

But $1 - t = t(1/t - 1) \leq t\lfloor 1/t \rfloor \leq t(1/t) = 1$ $(t \neq 0)$, so

$$1 = \lim_{t \to 0}(1 - t) \leq \lim_{t \to 0} h(t) \leq \lim_{t \to 0} 1 = 1.$$

Thus $g'(0) = \lim_{t \to 0} h(t) = 1$, which shows that g is differentiable at $x = 0$ with $g'(0) = 1$.

(c) Let ϵ be any positive number with $\epsilon < 1$. Is the function g Riemann integrable on $[\epsilon, 1]$? Justify your answer. For the extra point, is g Riemann integrable on $[0, 1]$?

Fix ϵ with $0 < \epsilon < 1$. The function $g(x)$ is continuous at all $x \in (\epsilon, 1]$, except possibly where $\lfloor 1/x \rfloor$ is not continuous, namely when $1/x \in \mathbb{N}$. So the only possible discontinuities of $g(x)$ in $(\epsilon, 1]$ are at $x = 1/n$ for $n \in \mathbb{N}$ with $1 \leq n \leq 1/\epsilon$. Since g is bounded and has only finitely many discontinuities on $[\epsilon, 1]$, we conclude that g is integrable on $[\epsilon, 1]$.

To confirm that g is integrable on $[0, 1]$, let $\epsilon \in (0, 1)$, and consider partitions P of $[0, 1]$ including the point ϵ. Since $0 \leq g(x) \leq \epsilon$ for $0 \leq x \leq \epsilon$, one sees that the contribution from the part of P lying in $[0, \epsilon]$ to both $L(P, g)$ and $U(P, g)$ is between 0 and ϵ^2. But we have already shown that g is integrable on $[\epsilon, 1]$. Thus, for each $\epsilon > 0$ one has

$$0 \leq \overline{\int_0^1} g(x)\,\mathrm{d}x - \underline{\int_0^1} g(x)\,\mathrm{d}x \leq \epsilon^2,$$

and hence

$$\overline{\int_0^1} g(x)\,\mathrm{d}x = \underline{\int_0^1} g(x)\,\mathrm{d}x.$$

We therefore conclude that g is integrable on $[0, 1]$.

Continued...

4. [15 points] Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is constant.

When $x \neq y$, we have

$$\left|\frac{f(x)-f(y)}{x-y}\right| \leq |x-y|,$$

and consequently, for all $y \in \mathbb{R}$ one has

$$\lim_{x \to y} \left|\frac{f(x)-f(y)}{x-y}\right| \leq \lim_{x \to y} |x-y| = 0.$$

The fact that the right hand limit is 0 shows that the left hand limit exists, and is equal to 0. Thus, using the arithmetic of limits, one sees that

$$\lim_{x \to y} \frac{f(x)-f(y)}{x-y} = 0.$$

This shows that for each $y \in \mathbb{R}$, the derivative $f'(y)$ exists and is equal to 0. We therefore conclude that f is constant.

5. [15 points] Suppose that f is a continuous mapping from $[0,1]$ to the metric space (X,d). Let (a_n) be a Cauchy sequence in $[0,1]$. Prove that $(f(a_n))$ is a Cauchy sequence in X.

Since (a_n) is Cauchy, given $\delta > 0$, there exists $N = N(\delta) \in \mathbb{N}$ such that whenever $n, m > N(\delta)$, one has $|a_n - a_m| < \delta$.

The mapping f is a continuous mapping from the compact set $[0,1]$ into X, and is therefore uniformly continuous on $[0,1]$. Thus, given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that, whenever $p, q \in [0,1]$ and $|p-q| < \delta$, one has $d(f(p), f(q)) < \epsilon$.

Take $N_0(\epsilon) = N(\delta(\epsilon))$, and combine the above two properties by taking $p = a_n$ and $q = a_m$. Thus, given $\epsilon > 0$, there exists $N_0 = N_0(\epsilon) \in \mathbb{N}$ having the property that whenever $n, m > N_0(\epsilon)$, one has $|a_n - a_m| < \delta(\epsilon)$, and hence $d(f(a_n), f(a_m)) < \epsilon$. This shows that the sequence $(f(a_n))$ is Cauchy in X.

6. [15 points] Suppose that $f : [0,2] \to \mathbb{R}$ is a differentiable function that satisfies $f'(0) = 1$ and $f'(2) = 10$. Suppose moreover that for all $x \geq 0$, the function f satisfies

$$f'(x) = (f(x))^3 - 1.$$

Show that there exists $z \in (0,2)$ such that $f(z) = 2$.

The function f is differentiable on $[0,2]$, and hence has the intermediate value property for derivatives. Since $f'(0) = 1 < 7 < 10 = f'(2)$, it follows that there exists $z \in (0,2)$ having the property that $f'(z) = 7$. But then $f(z)^3 - 1 = 7$, so that $f(z)^3 = 8$, whence $f(z) = 2$.

As an alternative, one can proceed using the intermediate value theorem. Since the function f is differentiable on $[0,2]$, it follows that f is continuous on $[0,2]$, whence the intermediate value theorem applies. But $f(0)^3 = 1 + f'(0) = 2$ and $f(2)^3 = 1 + f'(2) = 11$, so that $f(0) = 2^{1/3}$ and $f(2) = 11^{1/3}$. Since $f(0) = 2^{1/3} < 2 < 11^{1/3} = f(2)$, it follows that there exists $z \in (0,2)$ having the property that $f(z) = 2$.

End of examination.