PURDUE UNIVERSITY

Department of Mathematics

HONORS ALGEBRA
MA 45000 - SOLUTIONS

2nd November 2022   50 minutes

This paper contains **SIX** *questions.*
All SIX answers will be used for assessment.
Calculators, textbooks, notes and cribsheets are **not** *permitted in this examination.*

Do not turn over until instructed.

1. [4+4+4+4+4=20 points] Decide which of the following statements are necessarily true, and which may be false. Mark those which are true with "T", and those which are false with "F".

a. In the symmetric group S_7, no element has order 12.

Solution: FALSE (Consider $(1, 2, 3, 4)(5, 6, 7) \in S_7$, which has order $\mathrm{lcm}(4, 3) = 12$).

b. Any group of order 81 has an element of order 9.

Solution: FALSE (Consider the group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, which has order $3^4 = 81$, yet all of whose elements have order 1 or 3).

c. Suppose that G is a group of order 7^{2022}. Then the center of G is non-trivial.

Solution: TRUE (Corollary 2.11.4 from class shows that when G is a group of order p^n, with p prime and $n \geq 1$, then $Z(G) \neq \{e\}$).

d. The product of two odd permutations is an odd permutation.

Solution: FALSE (An odd permutation is a product of an odd number of transpositions, so the product of two odd permutations is the product of an even number of transpositions).

e. Let G be a simple group, and suppose that $\varphi : G \to G$ is a group homomorphism with $\varphi(g) \neq e$ for some $g \in G$. Then φ is an automorphism of G.

Solution: TRUE (Since $\varphi : G \to G$ is a homomorphism, we see that $\varphi(G)$ is a normal subgroup of G. But G is simple, so the only normal subgroups of G are $\{e\}$ and G. But $\varphi(G) \neq \{e\}$ since there exists $g \in G$ with $\varphi(g) \neq e$, so $\varphi(G) = G$. We conclude that φ is a bijective homomorphism, and hence an isomorphism from G to itself, and thus an automorphism).

2. [5+5+5+5=20 points]

(a) Suppose that $G_1, G_2, \ldots, G_n$ are groups. Define what is meant by the *(external) direct product* $G_1 \times G_2 \times \ldots \times G_n$.

Solution: The (external) direct product $G_1 \times G_2 \times \ldots \times G_n$ is the group

$$G = \{(g_1, g_2, \ldots, g_n) : g_i \in G_i \ (1 \leq i \leq n)\},$$

with the group operation defined by $(g_1, g_2, \ldots, g_n)(h_1, h_2, \ldots, h_n) = (g_1h_1, g_2h_2, \ldots, g_nh_n)$.

(b) Define what is meant by the statement that a group G is *simple*.

Solution: A group G is simple if it has no proper normal subgroups (thus the only normal subgroups of G are $\{e\}$ and G).

(c) Suppose that $\sigma \in S_n$. What is meant by the statement that σ is an *even permutation*?

Solution: The permutation $\sigma \in S_n$ is even if σ is the product of an even number of transpositions.

(d) Let G be a group of order $p^n m$, with p prime, $n \geq 1$ and $p \nmid m$. Define what is meant by a *Sylow p-subgroup* of G.

Solution: A Sylow p-subgroup of G is a subgroup P of G having order p^n (a maximal p-subgroup of G).

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3. [5+5+5+5=20 points] Consider the permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 1 & 3 & 9 & 7 & 6 & 5 & 8 \end{pmatrix}$.

   (a) Write σ as a product of disjoint cycles.

   Solution: We have $\sigma = (1, 2, 4, 3)(5, 9, 8)(6, 7)$.

   (b) What is the order of the element $\sigma \in S_9$?

   Solution: If σ is a product of cycles of length $m_1, \ldots, m_k$, then the order of σ is the least common multiple of $m_1, \ldots m_k$. Thus $\sigma = [4, 3, 2] = 12$.

   (c) Compute σ^{-1}.

   Solution: By reversing the cycles in σ, we see that

$$\sigma^{-1} = (1, 3, 4, 2)(5, 8, 9)(6, 7) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 8 & 7 & 6 & 9 & 5 \end{pmatrix}.$$

   (d) Compute the center of A_9.

   Solution: The group A_9 is simple, and hence the only normal subgroups of A_9 are $\{e\}$ and A_9. But $Z(G) \lhd G$ for all groups G, and thus $Z(A_9) = \{e\}$ or A_9. If $Z(A_9) = A_9$, then A_9 would be abelian, which is not the case $((1, 2)(1, 3) = (1, 3, 2) \neq (1, 2, 3) = (1, 3)(1, 2))$. Then we must have $Z(A_9) = \{e\}$, so that the center of A_9 is trivial.

4. [5+7=12 points] Let G be a group and put $A = G \times G$. Let T be the subgroup of A defined by $T = \{(g, g) : g \in G\}$.

   (a) Prove that $T \cong G$.

   Solution: Define the map $\varphi : G \to T$ by taking $g \mapsto (g, g)$. Then φ is plainly well-defined and surjective. Moreover, one has $\varphi(g) = \varphi(h)$ if and only if $(g, g) = (h, h)$, which holds if and only if $g = h$, and so φ is also injective. Finally, whenever $g, h \in G$, one has $\varphi(gh) = (gh, gh) = (g, g)(h, h) = \varphi(g)\varphi(h)$, so φ is a homomorphism. Thus, the map φ is an isomorphism, and so $T \cong G$.

   (b) Prove that $T \lhd A$ if and only if G is abelian.

   Solution: If G is abelian, then given any element $(a, a) \in T$, whenever $(g, h) \in A$ one has $(g, h)^{-1}(a, a)(g, h) = (g^{-1}ag, h^{-1}ah) = (g^{-1}ga, h^{-1}ha) = (a, a)$. Hence, for all $\gamma \in A$ one has $\gamma^{-1}T\gamma = T$, whence $T \lhd A$. If, on the other hand, one has $T \lhd A$, then for all $a, b \in G$ one has $(e, b)^{-1}(a, a)(e, b) \in A$, whence for some element $c \in G$ one has $(a, b^{-1}ab) = (c, c)$. Thus $c = a$ and $b^{-1}ab = c = a$. We therefore conclude that for all $a, b \in G$ one has $ab = ba$, which is to say that G is abelian. Thus $T \lhd A$ if and only if G is abelian.

5. [3+5=8 points] (a) State (any version of) the Fundamental Theorem of Finite Abelian Groups.

   Solution: Let G be a finite abelian group. Then G is isomorphic to a direct product of cyclic groups of the shape $\mathbb{Z}_{p_1^{\alpha_1}} \times \mathbb{Z}_{p_2^{\alpha_2}} \times \cdots \times \mathbb{Z}_{p_r^{\alpha_r}}$, where the p_i are (not necessarily distinct) prime numbers, the α_i are natural numbers, and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$.

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(b) Give a representative for every isomorphism class of abelian groups of order 500. For maximum credit, your answers should involve as few cyclic factors as possible.

Solution: Observe that $500 = 2^2 \cdot 5^3$. Then, by the classification theorem for finite abelian groups, representatives of the isomorphism classes of abelian groups of order 500 are

$$\begin{gathered}\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_5 \times \mathbb{Z}_{10} \times \mathbb{Z}_{10} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \cong \mathbb{Z}_{10} \times \mathbb{Z}_{50} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{125} \cong \mathbb{Z}_2 \times \mathbb{Z}_{250} \\ \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5 \cong \mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{20} \\ \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_{25} \cong \mathbb{Z}_5 \times \mathbb{Z}_{100} \\ \mathbb{Z}_4 \times \mathbb{Z}_{125} \cong \mathbb{Z}_{500}.\end{gathered}$$

6. [6+8+6=20 points] Suppose that p is a prime number with $p \geq 17$, and let G be a group of order $15p$.

(a) Suppose that G has a normal subgroup of order p. Show that G is abelian when $(15, p-1) = 1$.

Solution: Let A be a normal subgroup of G of order p. Since p is prime, the group A is cyclic, say $A = \langle a \rangle$. The group G/A has order 15, and since $15 = 5 \cdot 3$ with $3 \nmid (5-1)$, a theorem from class shows that G/A is also cyclic, say $G/A = \langle Ag \rangle$ for some $g \in G \setminus A$. By Lagrange's theorem, one has $Ag^{15} = (Ag)^{15} = A$, and hence $g^{15} = a^r$ for some integer r with $0 \leq r < p$. But $A \lhd G$, so $g^{-1}ag \in A$, whence $g^{-1}ag = a^i$ for some integer i with $1 \leq i \leq p-1$. But then $a = (a^{-r})a(a^r) = g^{-15}ag^{15} = a^{i^{15}}$, which shows that $i^{15} \equiv 1 \pmod{p}$. Fermat's Little Theorem shows that $i^{p-1} \equiv 1 \pmod{p}$, and so, since $(15, p-1) = 1$, we have $i \equiv 1 \pmod{p}$, whence $i = 1$, and thus $ag = ga$. Since every element of G takes the shape $a^u g^v$ for some integers u and v, and a and g commute, it follows that G is abelian.

(b) Prove that G has a normal subgroup of order p.

Solution: Since p divides $|G|$, it follows from Cauchy's theorem that G has an element of order p, and hence has a cyclic subgroup of order p, say $A = \langle a \rangle$. Suppose that B is another subgroup of order p distinct from A. Since $A \cap B \leq A$ and A has prime order, it follows from Lagrange's theorem that $A \cap B = A$ or $A \cap B = \{e\}$, whence $A \cap B = \{e\}$. But then $|AB| \geq |A| \cdot |B|/|A \cap B| \geq p^2$, and yet $AB \subseteq G$. This yields a contradiction, because $|G| = 15p < p^2$. Thus A is the only subgroup of G of order p, hence is fixed under conjugation by elements of G, and thus is a normal subgroup of G.

(c) Deduce that G is cyclic when $p \geq 17$ and $(15, p-1) = 1$.

Solution: From part (b), the group G has a normal subgroup, so that part (a) proves that G is abelian. But by Cauchy's theorem, since 3, 5 and p are all divisors of $|G|$, the group G contains elements of order 3, 5 and p, say a, b and c respectively. But then the order of the element abc is $15p$, since if r is any integer with $(abc)^r = e$, then $a^{5pr} = ((abc)^r)^{5p} = e$ whence $3|5pr$ implying that $3|r$, and similarly $b^{3pr} = ((abc)^r)^{3p} = e$ implying that $5|r$, and $c^{15r} = ((abc)^r)^{15} = e$ implying that $p|r$, all of which shows that $15p$ divides r and $o(abc) = 15p$. But then $|\langle abc \rangle| = |G| = 15p$, and so $G = \langle abc \rangle$ is cyclic.

End of examination.