HONORS ALGEBRA: HOMEWORK 9

Due noon Friday 4th November 2022

Herstein, section 2.11, page 106, Questions 15, 16, 18, 20, 23, 24.
Small groups (and the Sylow theorems):

SG.A In this problem, the letter p denotes a prime and n a natural number.

(a) Show that when $n \geq 2$ and G is a group of order p^n, then G is not simple.
(b) Show that when $p \neq 2$, then the group G cannot be simple if it has order $2p^n$ or $3p^n$ or $5p^n$.
(c) Show that when G is a group of order $4p^n$ $(p \neq 3)$ then G is not simple.

SG.B Let G be a group of order 56.

(a) Show that G has either 1 or 8 Sylow 7-subgroups, and either 1 or 7 Sylow 2-subgroups.
(b) By showing that, if there are 2 distinct Sylow 7-subgroups, they necessarily have trivial intersection, show that G cannot be simple.

SG.C By considering Sylow p-subgroups with p equal to 3 and 5, show that no group of order 30 is simple.

SG.D

(a) Show that no group of order 12, 24, 36 or 48 can be simple. [Hint: Recall homework question 16 from section 2.11].
(b) Let G be a simple group of order n, where $2 \leq n \leq 59$. Show that n is a prime number p and G is cyclic of order p.