HONORS ALGEBRA: SOLUTIONS TO HOMEWORK 6

3.3, Q2. Suppose that σ is a k-cycle. Then we can write σ in the form $(a_1, a_2, \ldots, a_k) = (a_k, a_1)(a_{k-1}, a_1)\ldots(a_2, a_1)$, for suitable distinct integers $a_1, \ldots, a_k$. Then σ is a product of $k-1$ transpositions, and hence is even when k is odd, and odd when k is even.

3.3, Q3. Suppose that $\sigma, \tau \in S_n$, and that σ is the product of k transpositions, and that τ is the product of l transpositions. Thus, say, $\sigma = \mu_1 \ldots \mu_k$ and $\tau = \nu_1 \ldots \nu_l$, with the μ_i and ν_i all transpositions. We thus see that $\tau^{-1} = \nu_l^{-1} \ldots \nu_1^{-1} = \nu_l \ldots \nu_1$, and hence $\tau^{-1}\sigma\tau = \nu_l \ldots \nu_1 \mu_1 \ldots \mu_k \nu_1 \ldots \nu_l$ is the product of $k+2l$ transpositions. Since $k+2l \equiv k \pmod 2$, we conclude that σ and $\tau^{-1}\sigma\tau$ have the same parity.

3.3, Q7. Write $\tau_1 = (2, 1, 3, 4, \ldots, n)$ and $\tau_2 = (n, n-1, \ldots, 3, 2, 1)$, and observe that $\tau_1\tau_2 = (1, 2, 3)$. Thus the 3-cycle $(1, 2, 3)$ is a product of two n-cycles. Let a_1, a_2, a_3 be distinct integers from $\{1, 2, \ldots, n\}$, and let σ be any permutation with $\sigma(1) = a_1$, $\sigma(2) = a_2$ and $\sigma(3) = a_3$. Then $\sigma\tau_i\sigma^{-1}$ is an n-cycle for $i = 1$ and 2, and moreover their product is $(\sigma\tau_1\sigma^{-1})(\sigma\tau_2\sigma^{-1}) = \sigma\tau_1\tau_2\sigma^{-1} = \sigma(1, 2, 3)\sigma^{-1} = (\sigma(1), \sigma(2), \sigma(3)) = (a_1, a_2, a_3)$. Thus all 3-cycles in S_n are the product of two n-cycles, and the latter product is necessarily even and hence lies in A_n. Since A_n is generated by 3-cycles, it follows that every element of A_n is a product of n-cycles.

6.1, Q1. Suppose that $n \geq 3$ and $\sigma \in Z(S_n) \setminus \{e\}$. Then for some $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have $\sigma(i) = j$. Since $n \geq 3$, there exists $k \in \{1, \ldots, n\}$ with $k \neq i$ and $k \neq j$. We take τ to be the transposition (j, k) and observe that $\tau\sigma\tau^{-1}(i) = k \neq j = \sigma(i)$, whence $\tau\sigma\tau^{-1} \neq \sigma$, and in particular one has $\tau\sigma \neq \sigma\tau$. We therefore deduce that whenever $\sigma \neq e$, then $\sigma \notin Z(S_n)$. Since e commutes with all elements of S_n, it follows that $Z(S_n) = \{e\}$, as required.

6.1, Q2. Suppose that $n \geq 4$ and $\sigma \in Z(A_n) \setminus \{e\}$. Then for some $i, j \in \{1, \ldots, n\}$ with $i \neq j$, we have $\sigma(i) = j$. Since $n \geq 4$, there exists $k, l \in \{1, \ldots, n\}$ with $k \neq l$ and neither k nor l equal to i or j. We take τ to be the 3-cycle (j, k, l). Since 3-cycles are even, we have $\tau \in A_n$. Moreover, one has $\tau\sigma\tau^{-1}(i) = k \neq j = \sigma(i)$, whence $\tau\sigma\tau^{-1} \neq \sigma$, and in particular one has $\tau\sigma \neq \sigma\tau$. We therefore deduce that whenever $\sigma \neq e$, then $\sigma \notin Z(A_n)$. Since e commutes with all elements of A_n, it follows that $Z(A_n) = \{e\}$, as required.

6.1, Q8. Suppose that $M \lhd N$ and $N \lhd G$. Then for all $a \in G$, and all $n \in N$, one has $a^{-1}na \in N$, whence $a^{-1}n = n_0a^{-1}$ for some $n_0 \in N$. But then $n^{-1}(aMa^{-1})n = (a^{-1}n)^{-1}M(a^{-1}n) = (n_0a^{-1})^{-1}Mn_0a^{-1} = a(n_0^{-1}Mn_0)a^{-1}$. But using $M \lhd N$, we see that $n_0^{-1}Mn_0 = M$, and hence $n^{-1}(aMa^{-1})n = aMa^{-1}$. Since this relation holds for all $n \in N$, we conclude that $aMa^{-1} \lhd N$ for all $a \in G$.