

**MA 59800-140 ANT ANALYTIC NUMBER THEORY:
INTRODUCTION TO THE CIRCLE METHOD AND ITS
APPLICATION, CRN 22328**

TREVOR D. WOOLEY

Class Meeting Times: Tuesdays and Thursdays 12:00 - 13:15

Class location/modality: Face-to-face in UNIV 317.

Credit Hours: 3 hours

Course web page: <https://www.math.purdue.edu/~twooley/2023ant/2023ant.html>

Course Brightspace page: <https://purdue.brightspace.com/d2l/home/709653>

Textbook: The course will be based on the instructor's lecture notes. Good texts for background reading and support are:

(a) R. C. Vaughan, *The Hardy-Littlewood method*, 2nd edn., Cambridge Tract No. 125, Cambridge University Press, 1997 [Condensed, but the best source in print; updated from the 1981 first edition.]

(b) H. Davenport, *Analytic methods for Diophantine equations and Diophantine inequalities*, Ann Arbor Publishers, Ann Arbor, 1962 or the LaTeXed version published by Cambridge University Press in 2005 [Friendlier for the basics, with material on general homogeneous cubics, but misses modern developments.]

(c) M. Nathanson, *Additive number theory. The classical bases*, GTM 164, Springer-Verlag, New York, 1996 [Pedestrian approach to the basics in which no corner is cut – good for getting started!]

Prerequisites: Elementary number theory and basic analysis.

Instructor: Prof. Trevor Wooley, twooley@purdue.edu

Location: 4.22 Math, Tel. 765-496-6439

Office Hours: Tu 16:30-17:30, W 13:00-14:00, Th 14:30-15:30 via Zoom or in person

***** ALL *** information for this course will be available from the course web-page. The use of Brightspace is only as a portal to this web-page. Use email to contact me, *NOT* Brightspace.**

Course Description: This course serves as an introduction to analytic number theory via the (Hardy-Littlewood) circle method. Background results from number theory and harmonic analysis will be reviewed as needed. Students already familiar with the basic elements of the circle method will acquire knowledge of more advanced topics, such as the use of smooth numbers, and the delta-function formulation of the method.

The (Hardy-Littlewood) circle method applies Fourier analysis to count rational or integral solutions of an equation or inequality in a manner respecting the inherent arithmetic. Developments in recent years have broadened its impact into additive combinatorics and discrete harmonic analysis beyond its more traditional role in quantitative arithmetic geometry.

We shall take as our central example Waring's problem – the problem of understanding the number of representations of an integer as the sum of a fixed number of k -th powers of positive integers. Our aims are twofold: (i) to understand the scope and limitations of the circle method, and (ii) to gain some facility to apply the method, so from time to time there will be technical material that we'll just cite rather than prove in any detail. This course is intended to be accessible to those without any background in analytic number theory, and to provide an introduction to some basic ideas in analytic number theory.

Learning outcomes: Students completing the course will: (i) acquire basic skills in an important theme of analytic number theory; (ii) gain experience applying analytic methods, and in particular harmonic analysis, to solve Diophantine problems; and (iii) be equipped to estimate exponential sums via the principal basic methods in the subject.

Course content:

- (i) Discussion of Weyl's inequality, Hua's Lemma, and the simplest treatment of Waring's problem. This provides an opportunity to discuss the key elements of the major arc analysis, that is, the singular integral and singular series, that together constitute the product of local densities.
- (ii) Smooth numbers – integers all of whose prime divisors are small – and their application in the circle method. Basic properties of the smooth numbers, including their distribution in arithmetic progression. Mean value estimates via efficient differencing, and new estimates of Weyl type.
- (iii) Application of smooth Weyl sums to equidistribution modulo 1, including fractional parts problems, and to upper bounds for the quantity $G(k)$ in Waring's problem.
- (v) The delta function variant of the circle method and its applications.

Assessment: Course credit will be based solely on six problem sets offered through the semester, posted on the course web-page:

<https://www.math.purdue.edu/~twooley/2023ant/2023ant.html>

Class participants can demonstrate engagement with the course by any written and/or in-class presentations featuring a reasonable subset of these problems. There are three levels of difficulty: short problems testing basic skill-sets, extended problems integrating the essential methods of the course, and more challenging problems for enthusiasts with detailed hints available on request. Working with other class members is permitted, but do write up the solutions individually by yourselves.

Homeworks should be submitted on paper in class by the indicated deadline or to the instructor's mailbox on the 4th floor of the Mathematics Building. Each homework is worth an equal amount of credit.

Purdue mandates that I include the following:

“Students who get at least 97% of the total points in this course are guaranteed an A+, 93% guarantees an A, 90% an A-, 87% a B+, 83% a B, 80% a B-, 77% a C+, 73% a C, 70% a C-, 67% a D+, 63% a D, and 60% a D-; for each of these grades, it's possible that at the end of the semester a somewhat lower percentage will be enough to get that grade”.

This is a graduate course, and you should imagine that a more nuanced approach will be taken in determining grades. In particular, it is likely that the numerical scores

required to achieve the grades listed will be lower than those indicated in the mandated text by at least 5-10%.

Attendance at lectures: Not mandatory, but strongly advised.

Arrangements for online aspects: We will use both Zoom and in-person options for Office Hours. The former should be available at:

<https://zoom.us>

Go to the upper right hand side of your screen to JOIN A MEETING, and then enter:

Meeting ID: XXX-XXX-XXXX

Password: XXXXX

This should take you through to the meeting space. I believe that you should be able to use the following short-cut, but if that does not work, do please use the Meeting ID and Password!

[https://purdue-
edu.zoom.us/j/XXXXXXXXXX?pwd=XXXXXXXXXXXXXXXXXXXXX](https://purdue-edu.zoom.us/j/XXXXXXXXXX?pwd=XXXXXXXXXXXXXXXXXXXXX)

[The meeting coordinates are not available from this public-facing page – go to the Brightspace website for the course, and download the syllabus from there to obtain the working coordinates for the zoom link]

Boilerplate Notes for Boilermakers:

This course will adhere to all of Purdue's standard regulations and requirements found here:

<https://catalog.purdue.edu/content.php?catoid=15&navoid=19004>).

These regulations are frequently updated, and what follows is a digest of the main points that may differ from the most recent updates in detail, but not in spirit. We will of course respect the most recent updates, as is required.

Academic guidance in the event you are quarantined/isolated. If you must miss class at any point in time during the semester, please reach out to me via email so that we can communicate about how you can maintain your academic progress. If you find yourself too sick to progress in the course, notify your adviser and notify me via email. We will make arrangements based on your particular situation. Please note that although individuals who test positive for COVID-19 are not guaranteed remote access to all course activities, materials, and assignments, we will attempt the best workaround that can be devised should you be impacted.

Attendance: This course follows Purdue's academic regulations regarding attendance, which states that students are expected to be present for every meeting of the classes in which they are enrolled. When conflicts or absences can be anticipated, such as for many University-sponsored activities and religious observations, the student should inform the instructor of the situation as far in advance as possible. For unanticipated or emergency absences when advance notification to the instructor is not possible, the student should contact the instructor as soon as possible by email or phone. When the student is unable to make direct contact with the instructor and is unable to leave word with the instructor's department because of circumstances beyond the student's control, and in cases falling under excused absence regulations, the student or the student's representative should contact or go to the Office of the Dean of Students website to complete appropriate forms for instructor notification. Under academic regulations, excused absences may be granted for cases of grief/bereavement, military service, jury duty, and parenting leave. There is now a Medically Excused Absence Policy for Students. For details, see the Academic Regulations and Student Conduct section of the University Catalog website. Guidance on class attendance related to COVID-19 are outlined in the Protect Purdue Pledge for Fall 2022 on the Protect Purdue website.

Protect Purdue: The Protect Purdue Plan, which includes the Protect Purdue Pledge, is campus policy and as such all members of the Purdue community must comply with the required health and safety guidelines. Required behaviors in this class include: staying home and contacting the Protect Purdue Health Center (496-INFO) if you feel ill or know you have been exposed to the virus.

Academic Integrity: Academic integrity is one of the highest values that Purdue University holds. Individuals are encouraged to alert university officials to potential breaches of this value by either emailing integrity@purdue.edu, or by calling 765-494-8778, or by contacting the Office of the Dean of Students (<https://www.purdue.edu/odos/>). While information may be submitted anonymously, the more information that is submitted provides the greatest opportunity for the university to investigate the concern. Purdue prohibits "dishonesty in connection with any University activity. Cheating, plagiarism,

or knowingly furnishing false information to the University are examples of dishonesty” (Section B.2.a of the Student Regulations

https://www.purdue.edu/studentregulations/student_conduct/regulations.html).

Furthermore, the University Senate has stipulated that “the commitment of acts of cheating, lying, and deceit in any of their diverse forms (such as the use of ghostwritten papers, the use of substitutes for taking examinations, the use of illegal cribs, plagiarism, and copying during examinations) is dishonest and must not be tolerated. Moreover, knowingly to aid and abet, directly or indirectly, other parties in committing dishonest acts is in itself dishonest.” Incidents of academic misconduct in this course will be addressed by the course instructor and referred to the Office of Student Rights and Responsibilities (OSRR) for review at the university level. Any violation of course policies as it relates to academic integrity will result minimally in a failing or zero grade for that particular assignment or test, and at the instructors discretion may result in a failing grade for the course. In addition, all incidents of academic misconduct will be forwarded to OSRR, where university penalties, including removal from the university, may be considered.

Boilermaker Honor Pledge: “As a Boilermaker pursuing academic excellence, I pledge to be honest and true in all that I do. Accountable together – we are Purdue.”

<https://www.purdue.edu/odos/osrr/honor-pledge/about.html>.

Nondiscrimination: Purdue University is committed to maintaining a community which recognizes and values the inherent worth and dignity of every person; fosters tolerance, sensitivity, understanding, and mutual respect among its members; and encourages each individual to strive to reach his or her own potential. In pursuit of its goal of academic excellence, the University seeks to develop and nurture diversity. The University believes that diversity among its many members strengthens the institution, stimulates creativity, promotes the exchange of ideas, and enriches campus life. More details are available on our course Brightspace table of contents, under University Policies.

Purdue’s nondiscrimination policy can be found at

https://www.purdue.edu/purdue/ea_eou_statement.php.

Basic Needs Security: Any student who faces challenges securing their food or housing and believes this may affect their performance in the course is urged to contact the Dean of Students for support. There is no appointment needed and Student Support Services is available to serve students 8 a.m.-5 p.m. Monday through Friday. Considering the significant disruptions caused by the current global crisis as it related to COVID-19, students may submit requests for emergency assistance from the Critical Needs Fund.

Academic Accommodation of Students with Disabilities: Purdue University strives to make learning experiences accessible to all participants. If you anticipate or experience physical or academic barriers based on disability, you are welcome to let me know so that we can discuss options. You are also encouraged to contact the Disability Resource Center at: drc@purdue.edu or by phone at 765-494-1247.

In this mathematics course accommodations are managed between the instructor, the student and DRC Testing Center. If you have been certified by the Disability Resource Center (DRC) as eligible for accommodations, you should contact your instructor to discuss your accommodations as soon as possible. Here are instructions for sending your Course Accessibility Letter to your instructor:

<https://www.purdue.edu/drc/students/course-accessibility-letter.php>

Mental Health: If you find yourself beginning to feel some stress, anxiety, and/or feeling slightly overwhelmed, try WellTrack at <https://purdue.welltrack.com/>. Sign in and find information and tools at your fingertips, available to you at any time. If you need support and information about options and resources, please see the Office of the Dean of Students, <http://www.purdue.edu/odos>, for drop-in hours (M-F, 8 am-5 pm). If you find yourself struggling to find a healthy balance between academics, social life, stress, etc. sign up for free one-on-one virtual or in-person sessions with a [Purdue Wellness Coach at RecWell](#). Student coaches can help you navigate through barriers and challenges toward your goals throughout the semester. Sign up is completely free and can be done on BoilerConnect. If you have any questions, please contact Purdue Wellness at evans240@purdue.edu. If you're struggling and need mental health services: Purdue University is committed to advancing the mental health and well-being of its students. If you or someone you know is feeling overwhelmed, depressed, and/or in need of mental health support, services are available. For help and to speak with a clinician, contact Counseling and Psychological Services (CAPS) at 765-494-6995 or by going to CAPS office on the second floor of the Purdue University Student Health Center (PUSH). For urgent situations after hours, on weekends and holidays, call 765-494-6995 to speak with a clinician. Please see <http://www.purdue.edu/caps/> for further information.

Commercial Note Taking in Classes: Notes taken in class are generally considered to be derivative works of the instructors presentations and materials, and they are thus subject to the instructors copyright in such presentations and materials. No individual is permitted to sell or otherwise barter notes, either to other students or to any commercial concern, for a course without the express written permission of the course instructor. See the Regulations on Student Conduct: Miscellaneous Conduct Regulations:

<http://catalog.purdue.edu/content.php?catoid=8&navoid=8208#miscellaneous-conductregulations>

Major Campus Emergency: In the event of a major campus emergency, course requirements, deadlines and grading percentages are subject to changes that may be necessitated by a revised semester calendar or other circumstances beyond the instructors control. Relevant changes to this course will be posted onto the course website or can be obtained by contacting the instructors or TAs via email or phone. You are expected to read your @purdue.edu email on a frequent basis.

EMERGENCY PREPAREDNESS LECTURE

As we begin this semester I want to take a few minutes and discuss emergency preparedness. Purdue University is a very safe campus and there is a low probability that a serious incident will occur here at Purdue. However, just as we receive a “safety briefing” each time we get on an aircraft, we want to emphasize our emergency procedures for evacuation and shelter in place incidents. Our preparedness will be critical IF an unexpected event occurs!

Emergency preparedness is your personal responsibility. Purdue University is actively preparing for natural disasters or human-caused incidents with the ultimate goal of maintaining a safe and secure campus. Let's review the following procedures:

For any emergency text or call 911.

There are more than 300 Emergency Telephones (aka blue lights) throughout campus that connect directly to the Purdue Police Department (PUPD). If you feel threatened or need help, push the button and you will be connected right away.

If we hear a fire alarm we will immediately evacuate the building and proceed to the sidewalk area near Primary location: between UNIV and BRNG; inclement weather: BRNG lobby.

- **Do not use the elevator.**

If we are notified of a Shelter in Place requirement for a tornado warning we will stop classroom or research activities and shelter in the lowest level of this building away from windows and doors. Our preferred location is the building interior or basement of BRNG Hall

If we are notified of a Shelter in Place requirement for a hazardous materials release we will shelter in our classroom shutting any open doors and windows.

If we are notified of a Shelter in Place requirement for an active threat such as a shooting we will shelter in a room that is securable, preferably without windows. Our preferred location is UNIV 317

Attached to the syllabus is an “Emergency Preparedness for Classrooms” sheet that provides additional preparedness information. Please review the sheet and the Emergency Preparedness website for additional emergency preparedness information.



EMERGENCY PREPAREDNESS SYLLABUS ATTACHMENT

EMERGENCY NOTIFICATION PROCEDURES are based on a simple concept – if you hear a fire alarm inside, proceed outside. If you hear a siren outside, proceed inside.

- **Indoor Fire Alarms** mean to stop class or research and immediately evacuate the building.
 - Proceed to your Emergency Assembly Area away from building doors. **Remain outside** until police, fire, or other emergency response personnel provide additional guidance or tell you it is safe to leave.
- **All Hazards Outdoor Emergency Warning Sirens** mean to immediately seek shelter (Shelter in Place) in a safe location within the closest building.
 - “Shelter in place” means seeking immediate shelter inside a building or University residence. This course of action may need to be taken during a tornado, an active threat including a shooting or a release of hazardous materials in the outside air. Once safely inside, find out more details about the emergency*. **Remain in place** until police, fire, or other emergency response personnel provide additional guidance or tell you it is safe to leave.

**In both cases, you should seek additional clarifying information by all means possible...Purdue Emergency Status page, text message, Twitter, Desktop Alert, Albertus Beacon, digital signs, email alert, TV, radio, etc....review the Purdue Emergency Warning Notification System multi-communication layers at http://www.purdue.edu/ehps/emergency_preparedness/warning-system.html*

EMERGENCY RESPONSE PROCEDURES:

- Review the **Emergency Procedures Guidelines**
https://www.purdue.edu/emergency_preparedness/flipchart/index.html
- Review the **Building Emergency Plan** (available on the Emergency Preparedness website or from the building deputy) for:
 - evacuation routes, exit points, and emergency assembly area
 - when and how to evacuate the building.
 - shelter in place procedures and locations
 - additional building specific procedures and requirements.

EMERGENCY PREPAREDNESS AWARENESS VIDEOS

- **"Run. Hide. Fight.®"** is a 6-minute active shooter awareness video that illustrates what to look for and how to prepare and react to this type of incident. See: https://www.youtube.com/watch?v=5mzI_5aj4Vs
(Link is also located on the EP website)

MORE INFORMATION

Reference the Emergency Preparedness web site for additional information:
https://www.purdue.edu/ehps/emergency_preparedness/