MA59800ANT ANALYTIC NUMBER THEORY. PROBLEMS 1

TO BE HANDED IN BY TUESDAY 24TH JANUARY 2023

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. (i) Suppose that $X > 0$ is large, and that x_1 and x_2 are two natural numbers with $x_1^2 + x_2^2 \leqslant X$. By making the trivial observation that $x_2^2 + x_1^2 = x_1^2 + x_2^2$, deduce that the number of integers n with $1 \leqslant n \leqslant X$ that are the sum of two squares of natural numbers is at most

$$\frac{1}{2}\lfloor\sqrt{X}\rfloor\left(\lfloor\sqrt{X}\rfloor + 1\right).$$

Hence deduce that *not all* large integers are the sum of two squares, and thus $G(2) \geqslant 3$.
(ii) Prove that when $k \geqslant 2$, one has $G(k) \geqslant k + 1$.

A2. (i) Show that when $a \in \mathbb{Z}$, then either $a^4 \equiv 0 \pmod{16}$ or $a^4 \equiv 1 \pmod{16}$.
(ii) Deduce that whenever n is an integer with $n \equiv 15 \pmod{16}$, then n cannot be the sum of 14 or fewer integral fourth powers.
(iii) Deduce that when r is a non-negative integer, then $31 \cdot 16^r$ is not the sum of 15 integral fourth powers, and hence $G(4) \geqslant 16$.

B3. When $\alpha \in \mathbb{R}$ and $X > 0$ is large, define the exponential sum

$$S(\alpha) = \sum_{1 \leqslant x \leqslant X} e(\alpha x).$$

(i) Show that when $p > 0$ and $p \neq 1$, one has

$$\int_0^1 |S(\alpha)|^p \, \mathrm{d}\alpha \ll 1 + X^{p-1}.$$

(ii) Prove that

$$\int_0^1 |S(\alpha)| \, \mathrm{d}\alpha \asymp \log(2X).$$

B4. (i) Suppose that $k \geqslant 2$. Write

$$f(\alpha) = \sum_{1 \leqslant x \leqslant X} e(\alpha x^k),$$

and recall the consequence of Hua's lemma showing that

$$\int_0^1 |f(\alpha)|^4 \, \mathrm{d}\alpha \ll X^{2+\varepsilon}.$$

Prove that for each positive number s, one has

$$\int_0^1 |f(\alpha)|^s \, \mathrm{d}\alpha \gg X^{s/2-\varepsilon}.$$

(ii) Show that when k is a positive integer and $s>0$, one has

$$\int_0^1 |f(\alpha)|^s \,\mathrm{d}\alpha \gg X^{s-k}.$$

(iii) Conclude that when $k\geqslant 2$ and $s>0$, one has

$$\int_0^1 |f(\alpha)|^s \,\mathrm{d}\alpha \gg X^{s/2-\varepsilon}+X^{s-k}.$$

B5. The forward difference operator Δ_1 is defined for any real-valued function ψ of the variable x via the relation

$$\Delta_1(\psi(x);h)=\psi(x+h)-\psi(x).$$

The j-fold forward difference operator is then defined for $j\geqslant 2$ inductively via the relation

$$\Delta_j(\psi(x);h_1,\ldots,h_j)=\Delta_1(\Delta_{j-1}(\psi(x);h_1,\ldots,h_{j-1});h_j).$$

Show that for $1\leqslant j\leqslant k$, one has

$$\Delta_j(x^k;h_1,\ldots,h_j)=\sum_{i_0,\ldots,i_j}\frac{k!}{i_0!\ldots i_j!}x^{i_0}h_1^{i_1}\ldots h_j^{i_j},$$

where the summation is over $i_0\geqslant 0, i_1\geqslant 1,\ldots,i_j\geqslant 1$ and $i_0+\cdots+i_j=k$. Hence deduce that

$$\Delta_j(x^k;h_1,\ldots,h_j)=h_1\ldots h_j p_j(x;h_1,\ldots,h_j),$$

where p_j is a polynomial in x of degree $k-j$ with leading coefficient $k!/(k-j)!$.

C6. Write

$$g(\alpha,\beta)=\sum_{1\leqslant x\leqslant X} e(\alpha x^3+\beta x).$$

Show that whenever $\alpha\in\mathbb{R}$, and $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$ and $|\alpha-a/q|\leqslant q^{-2}$, then one has

$$\int_0^1 |g(\alpha,\beta)|^4 \,\mathrm{d}\beta \ll X^{3+\varepsilon}(q^{-1}+X^{-1}+qX^{-3}).$$

C7. Suppose that (a_d) is a sequence of unimodular complex numbers. Show that

$$\int_0^1 \Bigl|\sum_{1\leqslant md^2\leqslant X} a_d e(\alpha m d^2)\Bigr|^{3/2} \,\mathrm{d}\alpha \ll X^{1/2+\varepsilon}.$$

Hint: Recall question B3 and consider dividing the range of summation for the variable d into dyadic intervals of the shape $(D,2D]$.