MA59800ANT ANALYTIC NUMBER THEORY. PROBLEMS 2

TO BE HANDED IN BY TUESDAY 7TH FEBRUARY 2022

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. (i) Let $\omega(r)$ be a complex-valued arithmetic function, and define $R_{s,k}(n;\omega)$ to be the number of solutions of the equation $n = x_1^k + \ldots + x_s^k$, with $x_i \in \mathbb{N}$, in which each solution $\mathbf{x}$ is counted with the weight $\omega(x_1)\omega(x_2)\ldots\omega(x_s)$. Show that

$$R_{s,k}(n;\omega) = \int_0^1 \Bigg(\sum_{1 \leqslant x \leqslant n^{1/k}} \omega(x) e(\alpha x^k) \Bigg)^s e(-n\alpha)\,\mathrm{d}\alpha.$$

(ii) When $\omega(r) = e^{-r^k/n}$, show that $R_{s,k}(n;\omega) = e^{-1} R_{s,k}(n;1)$, and conclude that

$$R_{s,k}(n;1) = e \int_0^1 \Bigg(\sum_{x=1}^{\infty} e^{-x^k/n} e(\alpha x^k) \Bigg)^s e(-n\alpha)\,\mathrm{d}\alpha.$$

A2. Let $n \in \mathbb{N}$ be large, and consider the problem of representing n in the shape

$$n = x_1^2 + x_2^2 + y_1^3 + y_2^3 + z_1^6 + z_2^6,$$

in which each variable x_i, y_i, z_i is a natural number.
(i) Show that one must have $1 \leqslant x_1, x_2 \leqslant n^{1/2}$, $1 \leqslant y_1, y_2 \leqslant n^{1/3}$ and $1 \leqslant z_1, z_2 \leqslant n^{1/6}$.
(ii) Write

$$f_k(\alpha) = \sum_{1 \leqslant x \leqslant n^{1/k}} e(\alpha x^k) \quad (k = 2, 3, 6).$$

Show that the number of representations of n in the above form is equal to

$$\int_0^1 f_2(\alpha)^2 f_3(\alpha)^2 f_6(\alpha)^2 e(-n\alpha)\,\mathrm{d}\alpha.$$

B3. Assume the notation of question A2. Show that

$$\int_0^1 |f_3(\alpha) f_6(\alpha)|^2\,\mathrm{d}\alpha = n^{1/2} + O(n^{1/3+\varepsilon})$$

and

$$\int_0^1 |f_2(\alpha) f_3(\alpha) f_6(\alpha)|^2\,\mathrm{d}\alpha \ll n^{1+\varepsilon}.$$

B4. Let $X \geqslant 1$ and $Y \geqslant 1$. Suppose that $\alpha \in \mathbb{R}$ satisfies the condition that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$.
(i) Suppose that $j \in \mathbb{N}$ satisfies $2^{1-j} X \geqslant q$. By applying Lemma 3.6, show that

$$\sum_{2^{-j} X < x \leqslant 2^{1-j} X} \min\left\{ XY/x, \|\alpha x\|^{-1} \right\} \ll XY \left(q^{-1} + 2^{-j} Y^{-1} + q(XY)^{-1} \right) \log(2q),$$

and hence deduce that

$$\sum_{q/2<x\leqslant X} \min\left\{XY/x, \|\alpha x\|^{-1}\right\} \ll XY\left(q^{-1}+Y^{-1}+q(XY)^{-1}\right)(\log(2qX))^2.$$

(ii) Show that when $1\leqslant x\leqslant q/2$, one has $\|\alpha x\|\geqslant 1/(2q)$. Hence deduce that

$$\sum_{1\leqslant x\leqslant q/2} \|\alpha x\|^{-1} \ll q\log(2q).$$

(iii) Combine the conclusions of (i) and (ii) to conclude that

$$\sum_{1\leqslant x\leqslant X} \min\left\{XY/x, \|\alpha x\|^{-1}\right\} \ll XY\left(q^{-1}+Y^{-1}+q(XY)^{-1}\right)(\log(2qX))^2.$$

B5. Define

$$v_1(\beta)=\int_0^X e(\beta\gamma^k)\,\mathrm{d}\gamma \quad \text{and} \quad v_2(\beta)=\sum_{1\leqslant m\leqslant X^k} \frac{1}{k}m^{1/k-1}e(m\beta).$$

(i) By estimating the expression

$$m^{1/k-1}e(\beta m)-\int_{m-1/2}^{m+1/2} t^{1/k-1}e(\beta t)\,\mathrm{d}t$$

for each $m\in\mathbb{N}$, establish that for all $\beta\in\mathbb{R}$, one has $v_1(\beta)-v_2(\beta)\ll 1+X|\beta|$.
(ii) Show that when n is an integer with $1\leqslant n\leqslant X^k$, then

$$J^*_{s,k}(n):=\int_{-1/2}^{1/2} v_2(\beta)^s e(-\beta n)\,\mathrm{d}\beta = \sum_{\substack{1\leqslant m_1,\ldots,m_s\leqslant n\\ m_1+\ldots+m_s=n}} k^{-s}m_1^{1/k-1}\cdots m_s^{1/k-1}.$$

Hence deduce that this analogue $J^*_{s,k}(n)$ of the singular integral satisfies $J^*_{s,k}(n)\gg n^{s/k-1}$.
C6. (i) Prove that the generating function

$$v_0(\beta)=\int_X^{2X} e(\beta\gamma^k)\,\mathrm{d}\gamma$$

satisfies the bound

$$v_0(\beta)\ll X(1+X^k|\beta|)^{-1}.$$

(ii) Prove that the singular integral associated with Waring's problem for s integral k-th powers converges absolutely for $s\geqslant 2$.
C7. Write $\tau(n)$ for the number of positive divisors of the natural number n. Show that, whenever $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$, then one has

$$\sum_{1\leqslant n\leqslant X} \tau(n)e(n\alpha) \ll X^{1+\varepsilon}\left(\frac{1}{q+X|q\alpha-a|}+X^{-1/2}+\frac{q+X|q\alpha-a|}{X}\right).$$