MA59800ANT ANALYTIC NUMBER THEORY. PROBLEMS 4

TO BE HANDED IN BY TUESDAY 7TH MARCH 2023

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. Consider the restricted sum of 14 biquadrates (which is to say, fourth powers)

$$x^4 + y^4 + (x+y)^4 + z_1^4 + z_2^4 + \ldots + z_{11}^4.$$

(i) Show that no integer n with $n \equiv 14 \pmod{16}$ can be represented by this restricted sum of 14 biquadrates.
(ii) Show that the congruence $x^4 + y^4 + (x+y)^4 + z_1^4 + z_2^4 + \ldots + z_{11}^4 \equiv n \pmod{16}$ is soluble with z_1 odd whenever $n \equiv r \pmod{16}$ and $1 \leqslant r \leqslant 13$.

A2. Apply the Cauchy-Davenport theorem to show that whenever $n \in \mathbb{Z}$, and p is an odd prime, then the congruence $x^4 + y^4 + (x+y)^4 + z_1^4 + z_2^4 + \ldots + z_{11}^4 \equiv n \pmod{p}$ is soluble with $(z_1, p) = 1$.

B3. Write

$$S(q,a) = \sum_{r=1}^{q} e\Big(\frac{ar^4}{q}\Big) \quad \text{and} \quad T(q,a) = \sum_{r=1}^{q}\sum_{s=1}^{q} e\Big(\frac{a(r^4 + s^4 + (r+s)^4)}{q}\Big).$$

Also, recall that as a consequence of Problems 3, Question C6, when $(a,q) = 1$, one has $S(q,a) \ll q^{3/4+\varepsilon}$.
(i) Prove the absolute convergence of the singular series

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} q^{-13} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q,a)^{11} T(q,a) e(-na/q).$$

(ii) Deduce that for some positive number C, one has $\mathfrak{S}(n) \geqslant \frac{1}{2}\prod_{p \leqslant C} \chi_p(n)$, where

$$\chi_p(n) = \lim_{h \to \infty} p^{-12h} M_n(p^h),$$

and $M_n(p^h)$ denotes the number of solutions of the congruence

$$x^4 + y^4 + (x+y)^4 + z_1^4 + z_2^4 + \ldots + z_{11}^4 \equiv n \pmod{p^h}$$

with $1 \leqslant x, y, z_i \leqslant p^h$.

B4. Let θ be a parameter satisfying $0 \leqslant \theta \leqslant 1/3$, and let $S(X,\theta)$ denote the number of solutions of the Diophantine equation

$$x^3 - y^3 = u_1^3 + u_2^3 - u_3^3 - u_4^3,$$

with $X < x, y \leqslant 2X$, and $1 \leqslant u_i \leqslant X^{1-\theta}$.
(i) Show that for any solution of the above equation with $x \neq y$, we have $|x - y| \leqslant X^{1-3\theta}$.

(ii) Using the substitution $y = x + h$, establish the upper bound $S(X, \theta) \leqslant XS_0 + 2S_1$, where S_0 denotes the number of solutions of the equation $u_1^3 + u_2^3 = u_3^3 + u_4^3$, with $1 \leqslant u_i \leqslant X^{1-\theta}$, and S_1 denotes the number of solutions of the equation

$$h(3x^2 + 3xh + h^2) = u_1^3 + u_2^3 - u_3^3 - u_4^3,$$

with $1 \leqslant u_i \leqslant X^{1-\theta}$, $1 \leqslant h \leqslant X^{1-3\theta}$ and $X < x \leqslant 2X$.

B5. (a) Let X and H be large real numbers, and define

$$F(\alpha) = \sum_{X \leqslant x \leqslant 2X} \sum_{1 \leqslant h \leqslant H} e(\alpha h(3x^2 + 3xh + h^2)).$$

By using a modification of Hua's lemma, show that

$$\int_0^1 |F(\alpha)|^4 \, \mathrm{d}\alpha \ll H^{3+\varepsilon} X^{2+\varepsilon}.$$

(b) Recall the notation of question B4 and put $H = X^{1-3\theta}$ and $Q = X^{1-\theta}$. Also, write

$$g(\alpha) = \sum_{1 \leqslant u \leqslant Q} e(\alpha u^3).$$

Show that

$$S_1 = \int_0^1 F(\alpha) |g(\alpha)|^4 \, \mathrm{d}\alpha,$$

and hence deduce that $S_1 \ll X^\varepsilon (H^3 X^2)^{1/4} Q^{9/4}$.

C6. (a) Combine your answers to questions B4 and B5 to deduce that

$$S(X, \theta) \ll X^\varepsilon (X^{3-2\theta} + X^{(7-9\theta)/2}),$$

and hence deduce that $S(X, 1/5) \ll X^{13/5+\varepsilon}$.

(b) Define

$$\mathcal{N}(N) = \mathrm{card}\{1 \leqslant n \leqslant N \ : \ n = x^3 + y^3 + z^3, \ x, y, z \in \mathbb{N}\}.$$

By considering sums of three cubes of the shape counted by $\mathcal{N}(N)$ with $X < x \leqslant 2X$ and $1 \leqslant y, z \leqslant X^{1-\theta}$, where $X = \frac{1}{3} N^{1/3}$, deduce that $\mathcal{N}(N) \gg N^{13/15-\varepsilon}$.

C7. Write

$$f(\alpha) = \sum_{1 \leqslant z \leqslant X} e(\alpha z^4) \quad \text{and} \quad g(\alpha) = \sum_{1 \leqslant x, y \leqslant X} e(\alpha(x^4 + y^4 + (x + y)^4)).$$

(i) Prove that

$$\int_0^1 |g(\alpha)^2 f(\alpha)^4| \, \mathrm{d}\alpha \ll X^{4+\varepsilon}.$$

(ii) Apply the Hardy-Littlewood circle method to prove that, whenever n is a sufficiently large positive integer satisfying $n \equiv r \pmod{16}$ with $1 \leqslant r \leqslant 13$, then n is represented as a sum of 14 positive integral biquadrates in the form

$$x^4 + y^4 + (x + y)^4 + z_1^4 + z_2^4 + \ldots + z_{11}^4 = n.$$