# MA59800ANT ANALYTIC NUMBER THEORY. PROBLEMS 5

TO BE HANDED IN BY TUESDAY 11TH APRIL 2023

**Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.**

**A1.** Suppose that $X \geqslant R \geqslant 2$ and $n \in \mathcal{A}(X,R)$. Show that for each prime number $p$ with $2 \leqslant p \leqslant X$, there is a unique decomposition $n = uv$, with $u \in \mathcal{A}(X,p)$ and $v$ an integer having all of its prime divisors $\pi$ satisfying the condition $p < \pi \leqslant R$.

**A2.** Let $P \geqslant M \geqslant R \geqslant 2$, and define the set

$$\mathcal{B}(M,\pi,R) = \{v \in \mathcal{A}(M\pi,R) : v > M,\ \pi|v \text{ and } \pi'|v \text{ implies } \pi' \geqslant \pi\}.$$

Here, the letters $\pi$ and $\pi'$ are intended to denote prime numbers. Show that when $v \in \mathcal{A}(P,R)$ satisfies $v > M$, there is a unique triple $(\pi,m,w)$ having the property that $v = mw$, $w \in \mathcal{A}(P/m,\pi)$ and $m \in \mathcal{B}(M,\pi,R)$.

**B3.** In this question, the function $\rho(\cdot)$ denotes the Dickman function.
(a) Show that, when $X$ is sufficiently large and $u > 1$, one has

$$\text{card}\{n \in [1,X] : n \text{ is } \textit{not } X^{1/u}\text{-smooth}\} = (1-\rho(u))X + O(X/\log X).$$

(b) Let $\eta$ be any positive number larger than $e^{-1/2} = 0.6065\ldots$. Show that when $n$ is large, the set $\{n - a : a \in \mathcal{A}(n,n^\eta)\}$ contains more than $n/2$ integers.
(c) Deduce that every large enough positive integer $n$ is the sum of two $n^\eta$-smooth integers.
[Open problem: establish the same conclusion for arbitrarily small positive values of $\eta$].

**B4.** Let $\eta$ be a positive number sufficiently small in terms of $k \geqslant 3$ and $\varepsilon$, and take $R$ to be a real number with $2 \leqslant R \leqslant P^\eta$. Define

$$f(\alpha;P,R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^k).$$

(a) Show that

$$\int_0^1 |f(\alpha;P,R)|^6 \,\mathrm{d}\alpha \ll P^{\lambda_3+\varepsilon},$$

where

$$\lambda_3 = 3 + \frac{2}{k}.$$

(b) Show that

$$\int_0^1 |f(\alpha;P,R)|^8 \,\mathrm{d}\alpha \ll P^{\lambda_4+\varepsilon},$$

where

$$\lambda_4 = 4 + \frac{5}{k} - \frac{2}{k^2}.$$

**B5.** (a) Show that the polynomial
$$\Psi(z;h;m)=m^{-4}\left((z+hm^4)^4-z^4\right),$$
has a root $z=-\frac{1}{2}hm^4$. Hence deduce that, as a polynomial in $z$, $h$ and $m$, the polynomial $\Psi(z;h;m)$ is divisible by $h$ and $2z+hm^4$.
(b) Write
$$F_1(\alpha)=\sum_{M<m\leqslant MR}\sum_{1\leqslant h\leqslant H}\sum_{1\leqslant z\leqslant P}e(\Psi(z;h;m)\alpha).$$
Prove that
$$\int_0^1|F_1(\alpha;h;m)|^2\,\mathrm{d}\alpha\ll(PHMR)^{1+\varepsilon}.$$

**C6.** Let $k$ be an even integer with $k\geqslant 4$. Put
$$\Psi(z;h;m)=m^{-k}\left((z+hm^k)^k-z^k\right),$$
and define
$$F_1(\alpha)=\sum_{M<m\leqslant MR}\sum_{1\leqslant h\leqslant H}\sum_{1\leqslant z\leqslant P}e(\Psi(z;h;m)\alpha).$$
(a) Prove that
$$\int_0^1|F_1(\alpha;h;m)|^2\,\mathrm{d}\alpha\ll(PHMR)^{1+\varepsilon}.$$
(b) Let $\theta$ be a positive number with $0<\theta\leqslant 1/k$, and write $M=P^\theta$, $Q=P/M$ and $H=PM^{-k}$. Using the notation from the course, show that when $R=P^\eta$, with $\eta>0$ sufficiently small, one has
$$S_3(P,R)\ll P^{1+\varepsilon}M^4Q^2+P^\varepsilon M^3\left(\int_0^1|F_1(\alpha)|^2\,\mathrm{d}\alpha\right)^{1/2}(S_4(Q,R))^{1/2}.$$
(c) Deduce that $S_3(P,P^\eta)\ll P^{\lambda_3+\varepsilon}$, where $\lambda_3=3+O(1/k^2)$ as $k\to\infty$.

**C7.** Adopt the notation of question B5 and let $k=4$. Take $\theta$ to be a real number with $1\leqslant\theta\leqslant 1/4$, and put $M=P^\theta$, $H=PM^{-4}$ and $Q=PM^{-1}$.
(a) By adapting the proof of Hua's Lemma, prove that
$$\int_0^1|F_1(\alpha)|^4\,\mathrm{d}\alpha\ll P^{2+\varepsilon}(HMR)^3.$$
(b) Taking $R=P^\eta$, with $\eta>0$ sufficiently small, show that
$$S_3(P,R)\ll P^{1+\varepsilon}M^4Q^2+P^\varepsilon M^3\left(\int_0^1|F_1(\alpha)|^3\,\mathrm{d}\alpha\right)^{1/3}(S_3(Q,R))^{2/3}.$$
(c) By using the conclusion of question B5, and optimising the choice of $\theta$, show that $S_3(P,R)\ll P^{\lambda_3+\varepsilon}$, where
$$\lambda_3=\frac{7+\sqrt{33}}{4}=3.1861406\ldots.$$

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