MA59800ANT ANALYTIC NUMBER THEORY. PROBLEMS 6

TO BE HANDED IN BY TUESDAY 25TH APRIL 2023

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

A1. For each natural number k with $k \geqslant 4$, and each natural number s, let $\Delta = \Delta_{s,k}$ denote the positive solution with $0 < \Delta < k$ of the equation

$$\Delta e^{\Delta/k} = ke^{1-2s/k}.$$

(a) Show that $\Delta < \frac{1}{2}k$ for $s > \frac{1}{2}(\frac{1}{2} + \log 2)k$.
(b) Suppose that $s = o(k)$ as $k \to \infty$. Show that $\Delta = k - s + \delta$, where

$$\delta = (1 + o(1))\frac{s^2}{4k}.$$

A2. Let $\eta > 0$ be sufficiently small in terms of t, $k \geqslant 4$ and ε, and write

$$f_\eta(\alpha) = \sum_{x \in \mathcal{A}(P,P^\eta)} e(\alpha x^k).$$

(a) Show that when $t > (\frac{1}{2} + \log 2)k + 1$, one has

$$\int_0^1 |f_\eta(\alpha)|^t \, \mathrm{d}\alpha \ll P^{t-k/2}.$$

(b) Suppose that $t = o(k)$ as $k \to \infty$. Show that

$$\int_0^1 |f_\eta(\alpha)|^t \, \mathrm{d}\alpha \ll P^{\frac{1}{2}t + (1+o(1))\frac{t^2}{16k}}.$$

B3. Let N be a large positive number, and write $P_k = N^{1/k}$ for each natural number k. Furthermore, put

$$f_k(\alpha) = \sum_{x \in \mathcal{A}(P_k,N^\eta)} e(\alpha x^k) \quad (k \in \mathbb{N}).$$

(a) Show that, when $k \geqslant 4$ and $t > (\frac{1}{2} + \log 2)k + 1$, one has

$$\int_0^1 |f_2(\alpha)^2 f_k(\alpha)^t| \, \mathrm{d}\alpha \ll f_2(0)^2 f_k(0)^t N^{\varepsilon - 1}.$$

(b) Show that, when $k \geqslant 4$ and $t > (\frac{1}{6} + \log \frac{6}{5})k + 1$, one has

$$\int_0^1 |f_2(\alpha)^2 f_3(\alpha)^2 f_k(\alpha)^t| \, \mathrm{d}\alpha \ll f_2(0)^2 f_3(0)^2 f_k(0)^t N^{\varepsilon - 1}.$$

(c) What do these mean value estimates imply about the problem of representing an integer as the sum of two squares and a number of k-th powers, or as the sum of two squares, two cubes, and a number of k-th powers? Brief remarks only here (no need to write a rigorous proof)!

B4. Let s and k be natural numbers with $k \geqslant 3$, and suppose that $\eta > 0$ is sufficiently small in terms of $\varepsilon > 0$ and s and k. Write $R = P^\eta$, and consider the sets

$$\mathcal{B}(Q,R) = \{m \in [1,Q] \cap \mathbb{Z} : p|m \text{ implies } \sqrt{R} < p \leqslant R\}$$

and

$$\mathcal{C} = \{n \in [1,P] \cap \mathbb{Z} : n = uw, \text{ where } 1 \leqslant u \leqslant \sqrt{R} \text{ and } w \in \mathcal{B}(P/\sqrt{R}, R)\}.$$

Also, define the exponential sum

$$h(\alpha) = \sum_{x \in \mathcal{C}} e(\alpha x^k).$$

(a) Show that each element of $\mathcal{C}$ is uniquely represented in the form $n = uw$ with u and w as described in the definition of $\mathcal{C}$.

(b) Show that for each natural number s, one has

$$\int_0^1 |h(\alpha)|^{2s}\,\mathrm{d}\alpha \leqslant \int_0^1 \biggl| \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^k) \biggr|^{2s} \mathrm{d}\alpha.$$

C5. Adopt the same notation as in question B4, and write

$$S(q,a) = \sum_{r=1}^{q} e(ar^k/q) \quad \text{and} \quad v(\beta) = \int_0^{\sqrt{R}} e(\beta\gamma^k)\,\mathrm{d}\gamma.$$

(a) Prove that $\mathrm{card}(\mathcal{C}) \gg P(\log P)^{-3/\eta}$.

(b) Suppose that $\alpha \in \mathbb{R}$, and $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a,q) = 1$, $1 \leqslant q \leqslant R^{1/6}$ and $|q\alpha - a| \leqslant R^{1/6}P^{-k}$. Show that

$$h(\alpha) = q^{-1}S(q,a) \sum_{w \in \mathcal{B}(P/\sqrt{R},R)} v((\alpha - a/q)w^k) + O(PR^{-1/6}).$$

(c) Deduce, under the hypotheses of part (a), that one has the estimate

$$h(\alpha) \ll P(\log P)(q + P^k|q\alpha - a|)^{\varepsilon - 1/k}.$$