NUMBER THEORY: HOMEWORK 1

DUE WEDNESDAY 30TH AUGUST 2023

1. (i) Prove that for every natural number $n \geqslant 3$, one has $(n-2)|(n^3-8)$.
(ii) Suppose that n is a natural number exceeding 1. Prove that

$$(n^2-1, n^4+n) = n+1.$$

2. (i) Let a and b be integers. Show that $9|(10a+b)$ if and only if $9|(a+b)$, and hence deduce that an integer n is divisible by 9 if and only if the sum of its base-10 digits is divisible by 9.
(ii) Let a and b be integers. Show that $33|(100a+b)$ if and only if $33|(a+b)$, and hence deduce that an integer n is divisible by 33 if and only if the sum of its base-100 digits is divisible by 33.
(iii) Let a and b be integers. Show that $37|(1000a+b)$ if and only if $37|(a+b)$, and hence deduce that an integer n is divisible by 37 if and only if the sum of its base-1000 digits is divisible by 37.

3. Let the conventional base 10 expansion of the integer n be $n_k n_{k-1} \dots n_1 n_0$, so that

$$n = 10^k n_k + 10^{k-1} n_{k-1} + \dots + n_0 \quad \text{with} \quad n_i \in \{0, 1, \dots, 9\}.$$

Let m be the integer with base 10 expansion $n_k n_{k-1} \dots n_1$, so that

$$m = 10^{k-1} n_k + 10^{k-2} n_{k-1} + \dots + n_1.$$

(i) Show that $2n$ (and hence also n) is divisible by 19 if and only if $m + 2n_0$ is divisible by 19, thereby providing a test for divisibility by 19.
(ii) Show that n is divisible by 7 if and only if $m + 5n_0$ is divisible by 7, thereby providing a test for divisibility by 7.

4. Let n be a natural number.
(i) Prove that $(n!-1, (n+1)!-1) = 1$.
(ii) Prove that $(n!+1, (n+1)!+1) = 1$.

5. By considering the binomial coefficient $\binom{n}{k}$, prove that the product of k consecutive integers is always divisible by $k!$.

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