NUMBER THEORY: HOMEWORK 10

TO BE HANDED IN BY WEDNESDAY 15TH NOVEMBER 2023

1. (a) Prove that

$$\sum_{1\leqslant n\leqslant x} \frac{\sigma(n)}{n^2} = \frac{1}{6}\pi^2 \log x + O(1),$$

where $\sigma(n)$ denotes the sum of divisors of n.
(b) Write down a relation between the Euler function $\phi(n)$ and the Möbius function $\mu(n)$. Hence, or otherwise, obtain an expression, analogous to that given above, for

$$\sum_{1\leqslant n\leqslant x} \frac{\phi(n)}{n^2}.$$

2. (a) Using multiplicativity, prove that

$$\sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} = \prod_p (1+1/p^2).$$

(b) Hence deduce that

$$\sum_{1\leqslant d\leqslant x} \frac{\mu^2(d)}{d^2} = \frac{15}{\pi^2} + O\Big(\frac{1}{x}\Big).$$

Here, you may assume without proof that

$$\zeta(2) = \prod_p (1-1/p^2)^{-1} = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \prod_p (1-1/p^4)^{-1} = \frac{\pi^4}{90}.$$

3. Recall that, when f is an arithmetic function, then

$$\sum_{\substack{1\leqslant a\leqslant n\\ (a,n)=1}} f(a) = \sum_{d|n} \mu(d) \sum_{\substack{1\leqslant a\leqslant n\\ d|a}} f(a).$$

Let $\alpha(n)$ denote the arithmetical function defined by $\alpha(n) = \prod_{\substack{1\leqslant a\leqslant n\\ (a,n)=1}} a$. Prove that

$$\alpha(n) = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}.$$