NUMBER THEORY: HOMEWORK 3

TO BE HANDED IN BY WEDNESDAY 13TH SEPTEMBER 2023

1. (i) What are the last three digits in the ordinary decimal expansion of 79^{7201}? What are the last two digits in the expansion of 5^{2023}?

(ii) Show that $23|(2^{2n+5} - 3^{3n+2})$ for all $n \in \mathbb{N}$.

2. (i) Show that $x^3 \equiv 2 \pmod 7$ has no solution. Deduce that $x^3 - 2y^3 = 0$ has no non-zero solution in integers x and y. Deduce that $\sqrt[3]{2}$ is irrational.

(ii) Show that the equation $x^3 - 2y^3 + 7z^3 = 0$ has no non-zero solution in integers x, y and z.

3. (i) Show that for every natural number n, the least prime **not** dividing n is no larger than $n + 1$. [Hint: consider Euclid's proof that there are infinitely many primes]

(ii) Prove that for each prime number q and natural number n, one has $q^n \geqslant n + 1$.

(iii) Deduce that if p is the least prime **not** dividing n, then $p - 1$ divides n^n.

(iv) Prove that when n is a natural number, then the least prime **not** dividing n is the smallest prime divisor of $n^{n^n} - 1$.

(v) Let p_k be the k-th smallest prime number, so that $p_1 = 2$, $p_2 = 3$, etc. Prove that p_{k+1} is the smallest prime divisor of $n^{n^n} - 1$, where $n = p_1 p_2 \dots p_k$.