

NUMBER THEORY: HOMEWORK 3

TO BE HANDED IN BY WEDNESDAY 13TH SEPTEMBER 2023

1. (i) What are the last three digits in the ordinary decimal expansion of 79^{7201} ? What are the last two digits in the expansion of 5^{2023} ?
(ii) Show that $23|(2^{2n+5} - 3^{3n+2})$ for all $n \in \mathbb{N}$.
2. (i) Show that $x^3 \equiv 2 \pmod{7}$ has no solution. Deduce that $x^3 - 2y^3 = 0$ has no non-zero solution in integers x and y . Deduce that $\sqrt[3]{2}$ is irrational.
(ii) Show that the equation $x^3 - 2y^3 + 7z^3 = 0$ has no non-zero solution in integers x, y and z .
3. (i) Show that for every natural number n , the least prime **not** dividing n is no larger than $n + 1$. [Hint: consider Euclid's proof that there are infinitely many primes]
(ii) Prove that for each prime number q and natural number n , one has $q^n \geq n + 1$.
(iii) Deduce that if p is the least prime **not** dividing n , then $p - 1$ divides n^n .
(iv) Prove that when n is a natural number, then the least prime **not** dividing n is the smallest prime divisor of $n^{n^n} - 1$.
(v) Let p_k be the k -th smallest prime number, so that $p_1 = 2, p_2 = 3$, etc. Prove that p_{k+1} is the smallest prime divisor of $n^{n^n} - 1$, where $n = p_1 p_2 \dots p_k$.

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