NUMBER THEORY: HOMEWORK 5

TO BE HANDED IN BY WEDNESDAY 27TH SEPTEMBER 2023

1. Let $f(x) = x^2 - x$ throughout.

(a) Show that for every prime number p and every positive integer k, the congruence $f(x) \equiv 0 \pmod{p^k}$ has precisely 2 solutions.

(b) Let m be a natural number, and let r denote the number of distinct prime numbers dividing m. Show that the congruence $f(x) \equiv 0 \pmod{m}$ has precisely 2^r solutions.

2. (a) Prove that if p is prime, $(a, p) = 1$ and $(n, p - 1) = 1$, then $x^n \equiv a \pmod{p}$ has exactly one solution.

(b) Show that when $(n, p - 1) = d$, then $x^n \equiv 1 \pmod{p}$ has precisely d solutions.

3. (a) Find a solution of $x^4 + x + 1 \equiv 0 \pmod{3^3}$.

(b) Show that $x^2 + 6x + 31 \equiv 0 \pmod{121}$ has no solutions.

4. (a) Prove that if a belongs to h modulo a prime p (i.e. a has order h modulo p), and if h is even, then $a^{h/2} \equiv -1 \pmod{p}$.

(b) Suppose that p is odd and a belongs to h modulo p^k for some integer $k \geqslant 2$. Is it necessarily the case that $a^{h/2} \equiv -1 \pmod{p^k}$?

5. Show that $x^p \equiv x \pmod{p^j}$ has precisely p solutions modulo p^j for every prime power p^j.