

## NUMBER THEORY: HOMEWORK 5

TO BE HANDED IN BY WEDNESDAY 27TH SEPTEMBER 2023

**1.** Let  $f(x) = x^2 - x$  throughout.

(a) Show that for every prime number  $p$  and every positive integer  $k$ , the congruence  $f(x) \equiv 0 \pmod{p^k}$  has precisely 2 solutions.

(b) Let  $m$  be a natural number, and let  $r$  denote the number of distinct prime numbers dividing  $m$ . Show that the congruence  $f(x) \equiv 0 \pmod{m}$  has precisely  $2^r$  solutions.

**2.** (a) Prove that if  $p$  is prime,  $(a, p) = 1$  and  $(n, p - 1) = 1$ , then  $x^n \equiv a \pmod{p}$  has exactly one solution.

(b) Show that when  $(n, p - 1) = d$ , then  $x^n \equiv 1 \pmod{p}$  has precisely  $d$  solutions.

**3.** (a) Find a solution of  $x^4 + x + 1 \equiv 0 \pmod{3^3}$ .

(b) Show that  $x^2 + 6x + 31 \equiv 0 \pmod{121}$  has no solutions.

**4.** (a) Prove that if  $a$  belongs to  $h$  modulo a prime  $p$  (i.e.  $a$  has order  $h$  modulo  $p$ ), and if  $h$  is even, then  $a^{h/2} \equiv -1 \pmod{p}$ .

(b) Suppose that  $p$  is odd and  $a$  belongs to  $h$  modulo  $p^k$  for some integer  $k \geq 2$ . Is it necessarily the case that  $a^{h/2} \equiv -1 \pmod{p^k}$ ?

**5.** Show that  $x^p \equiv x \pmod{p^j}$  has precisely  $p$  solutions modulo  $p^j$  for every prime power  $p^j$ .

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