

## NUMBER THEORY: HOMEWORK 8

TO BE HANDED IN BY WEDNESDAY 25TH OCTOBER 2023

1. Calculate the symbols  $\left(\frac{264}{173}\right)$ ,  $\left(\frac{2019}{4987}\right)$ ,  $\left(\frac{187}{389}\right)$ .
2. (a) Determine the odd prime numbers  $p$  for which 5 is a quadratic residue modulo  $p$ .  
(b) By considering the polynomial  $x^2 - 5$ , and applying a variant of Euclid's proof, show that there are infinitely many primes of the shape  $5k + 4$ .
3. Let  $p$  be an odd prime number. Determine the primes  $p$  for which  $-7$  is a quadratic residue modulo  $p$ .
4. (a) Show that  $\left(\frac{3}{p}\right) = -1$  whenever  $p \equiv 5 \pmod{12}$ .  
(b) Suppose that  $p = 2^{2^n} + 1$  is a prime number. Show that 3 is a primitive root modulo  $p$ .
5. Use the following strategy to prove that there are no integers  $x, y$  satisfying the equation  $y^2 = x^3 + 45$ .  
(a) Show that if  $(x, y)$  were to satisfy this equation, then  $x \equiv 7 \pmod{8}$  or  $x \equiv 3 \pmod{8}$ .  
(b) If  $x \equiv 7 \pmod{8}$ , rewrite the equation as

$$y^2 - 2 \cdot 3^2 = (x + 3)(x^2 - 3x + 9).$$

Prove that  $x^2 - 3x + 9$  must be divisible by a prime  $p \equiv \pm 3 \pmod{8}$ , and derive a contradiction.

- (c) Deal with the second case in which  $x \equiv 3 \pmod{8}$  by writing  $y^2 = x^3 + 45$  as

$$y^2 - 2 \cdot 6^2 = (x - 3)(x^2 + 3x + 9),$$

and proceeding similarly.

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