

NUMBER THEORY: HOMEWORK 9

TO BE HANDED IN BY WEDNESDAY 1ST NOVEMBER 2023

1. (a) Prove that for all natural numbers n , one has $\sum_{d|n} \mu^2(d) = 2^{\omega(n)}$.
(b) Prove that for all natural numbers n , one has $\sum_{d|n} \mu(d)\tau(d) = (-1)^{\omega(n)}$.

2. (a) Show that

$$\sum_{a=1}^n a^3 = \frac{1}{4}n^2(n+1)^2 = \left(\sum_{a=1}^n a\right)^2.$$

- (b) Prove that for all natural numbers n , one has $\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^2$.

3. Let f be an arithmetic function.

(a) Prove that when a and n are positive integers, then

$$\sum_{d|(a,n)} \mu(d) = \begin{cases} 1, & \text{when } (a,n) = 1, \\ 0, & \text{when } (a,n) > 1. \end{cases}$$

(b) Deduce that

$$\sum_{\substack{1 \leq a \leq n \\ (a,n)=1}} f(a) = \sum_{d|n} \mu(d) \sum_{\substack{1 \leq a \leq n \\ d|a}} f(a).$$

(c) Prove that when $n > 1$, one has

$$\sum_{\substack{1 \leq a \leq n \\ (a,n)=1}} 1 = \varphi(n) \quad \text{and} \quad \sum_{\substack{1 \leq a \leq n \\ (a,n)=1}} a = \frac{1}{2}n\varphi(n).$$

4. Let $s(n)$ denote the arithmetic function, referred to as the squarefree kernel of n , defined by putting $s(n) = \prod_{p|n} p$.

(a) Show that $s(n)$ is a multiplicative function of n

(b) By applying Möbius inversion, find an arithmetic function $f(n)$ having the property that $s(n) = \sum_{d|n} f(d)$, and write $f(n)$ explicitly in terms of familiar arithmetic functions such as $\tau(n)$, $\varphi(n)$, $\mu(n)$, and so on.

5. (a) Suppose that $a(n)$ and $b(n)$ are multiplicative functions. Show that the arithmetic function $c(n) = \sum_{d|n} a(n/d)b(d)$ is also multiplicative.

(b) Show that $\sigma(n) = \sum_{d|n} \varphi(n/d)\tau(d)$.

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