NUMBER THEORY: HOMEWORK 9

TO BE HANDED IN BY WEDNESDAY 1ST NOVEMBER 2023

1. (a) Prove that for all natural numbers n, one has $\sum_{d|n}\mu^2(d)=2^{\omega(n)}$.
(b) Prove that for all natural numbers n, one has $\sum_{d|n}\mu(d)\tau(d)=(-1)^{\omega(n)}$.

2. (a) Show that

$$\sum_{a=1}^{n}a^3=\tfrac{1}{4}n^2(n+1)^2=\left(\sum_{a=1}^{n}a\right)^2.$$

(b) Prove that for all natural numbers n, one has $\sum_{d|n}\tau(d)^3=\left(\sum_{d|n}\tau(d)\right)^2$.

3. Let f be an arithmetic function.
(a) Prove that when a and n are positive integers, then

$$\sum_{d|(a,n)}\mu(d)=\begin{cases}1, & \text{when } (a,n)=1,\\ 0, & \text{when } (a,n)>1.\end{cases}$$

(b) Deduce that

$$\sum_{\substack{1\leqslant a\leqslant n\\(a,n)=1}}f(a)=\sum_{d|n}\mu(d)\sum_{\substack{1\leqslant a\leqslant n\\ d|a}}f(a).$$

(c) Prove that when $n>1$, one has

$$\sum_{\substack{1\leqslant a\leqslant n\\(a,n)=1}}1=\varphi(n)\quad\text{and}\quad\sum_{\substack{1\leqslant a\leqslant n\\(a,n)=1}}a=\tfrac{1}{2}n\varphi(n).$$

4. Let $s(n)$ denote the arithmetic function, referred to as the squarefree kernel of n, defined by putting $s(n)=\prod_{p|n}p$.
(a) Show that $s(n)$ is a multiplicative function of n
(b) By applying Möbius inversion, find an arithmetic function $f(n)$ having the property that $s(n)=\sum_{d|n}f(d)$, and write $f(n)$ explicity in terms of familiar arithmetic functions such as $\tau(n)$, $\varphi(n)$, $\mu(n)$, and so on.

5. (a) Suppose that $a(n)$ and $b(n)$ are multiplicative functions. Show that the arithmetic function $c(n)=\sum_{d|n}a(n/d)b(d)$ is also multiplicative.
(b) Show that $\sigma(n)=\sum_{d|n}\varphi(n/d)\tau(d)$.