SOLUTIONS TO HOMEWORK 1

1. (i) When $n \geqslant 2$, one has $n^3 - 8 = (n-2)(n^2+2n+4)$, and so $n-2$ divides $n^3 - 8$, as required.
(ii) When $n \geqslant 2$, one has $(n^2-1, n^4+n) = (n^2-1, (n^2-1)(n^2+1)+n+1)$, and thus $(n^2-1, n^4+n) = ((n+1)(n-1), n+1) = (n+1)(n-1, 1) = n+1$.

2. (i) One has $9|(10a+b)$ if and only if $9|(10a+b-9a)$, or equivalently $9|(a+b)$. Write $n = 10^k n_k + 10^{k-1}n_{k-1} + \ldots + n_0$ in the ordinary base-10 expansion. Using the above conclusion, one finds that $9|n$ if and only if

$$9|(10^{k-1}n_k + \ldots + 10n_2 + n_1 + n_0),$$

or equivalently $9|(10^{k-2}n_k + \ldots + 10n_3 + n_2 + (n_1 + n_0))$, and so on. Thus, by induction, one sees that $9|n$ if and only if $9|(n_k + n_{k-1} + \ldots + n_0)$, as required.
(ii) One has $33|(100a+b)$ if and only if $33|(100a+b-3(33a))$, or equivalently $33|(a+b)$. Write $n = 100^k n_k + 100^{k-1}n_{k-1} + \ldots + n_0$ in the ordinary base-100 expansion. Using the above conclusion, one finds that $33|n$ if and only if $33|(100^{k-1}n_k + \ldots + 100n_2 + n_1 + n_0)$, or equivalently

$$33|(100^{k-2}n_k + \ldots + 100n_3 + n_2 + (n_1 + n_0)),$$

and so on. Thus, one sees that $33|n$ if and only if $33|(n_k + n_{k-1} + \ldots + n_0)$, as required.
(iii) One has $37|(1000a+b)$ if and only if $37|(1000a+b-27(37a))$, or equivalently $37|(a+b)$. Write $n = 1000^k n_k + 1000^{k-1}n_{k-1} + \ldots + n_0$ in the ordinary base-1000 expansion. Using the above conclusion, one finds that $37|n$ if and only if $37|(1000^{k-1}n_k + \ldots + 1000n_2 + n_1 + n_0)$, or equivalently

$$37|(1000^{k-2}n_k + \ldots + 1000n_3 + n_2 + (n_1 + n_0)),$$

and so on. Thus, one sees that $37|n$ if and only if $37|(n_k + n_{k-1} + \ldots + n_0)$, as required.

3. (i) Since 2 and 19 are coprime, one finds that $n = 10m + n_0$ is divisible by 19 if and only if $2n = 20m + 2n_0$ is divisible by 19. But the latter holds if and only if $20m + 2n_0 - 19m = m + 2n_0$ is divisible by 19. Thus $19|n$ if and only if $m + 2n_0$ is divisible by 19, as required.
(ii) Since 5 and 7 are coprime, one finds that $n = 10m + n_0$ is divisible by 7 if and only if $5n = 50m + 5n_0$ is divisible by 7. But the latter holds if and only if $50m + 5n_0 - 7(7m) = m + 5n_0$ is divisible by 7. Thus $7|n$ if and only if $m + 5n_0$ is divisible by 7, as required.

4. (i) One has $(n!-1, (n+1)!-1) = (n!-1, ((n+1)!-1) - (n+1)(n!-1)) = (n!-1, n)$. But $(n!-1, n) = (n!-1-n\cdot(n-1)!, n) = (-1, n) = 1$, and so $(n!-1, (n+1)!-1) = 1$, as required.

(ii) One has $(n!+1,(n+1)!+1)=(n!+1,((n+1)!+1)-(n+1)(n!+1))=(n!+1,-n)$. But $(n!-1,-n)=(n!-1-n\cdot(n-1)!,-n)=(-1,-n)=1$, and so $(n!+1,(n+1)!+1)=1$, as required.

5. If the k consecutive integers in question contain 0, then this conclusion is trivial. Also, when all k integers are negative, then their product is equal to $(-1)^k$ multiplied by the product of k consecutive positive integers, and thus there is no loss of generality in restricting to the case of k consecutive positive integers. Whenever $k,n\in\mathbb{N}$ satisfy $k\leqslant n$, one has

$$\frac{n(n-1)\cdots(n-k+1)}{k!}=\binom{n}{k}\in\mathbb{N},$$

and hence $k!$ divides $n(n-1)\cdots(n-k+1)$. Then the product of any k positive integers is divisible by $k!$, and this completes the proof.