

SOLUTIONS TO HOMEWORK 10

1. (a) We have $\sigma(n) = \sum_{d|n} d$, and hence

$$\begin{aligned} \sum_{1 \leq n \leq x} \frac{\sigma(n)}{n^2} &= \sum_{1 \leq n \leq x} \sum_{d|n} \frac{d}{n^2} = \sum_{1 \leq d \leq x} \sum_{1 \leq m \leq x/d} \frac{d}{(md)^2} \\ &= \sum_{1 \leq d \leq x} \frac{1}{d} \sum_{1 \leq m \leq x/d} \frac{1}{m^2} = \sum_{1 \leq d \leq x} \frac{1}{d} \left(\frac{\pi^2}{6} + O(d/x) \right) \\ &= \frac{\pi^2}{6} \sum_{1 \leq d \leq x} \frac{1}{d} + O\left(\frac{1}{x} \sum_{1 \leq d \leq x} 1 \right) \\ &= \frac{\pi^2}{6} \log x + O(1). \end{aligned}$$

(b) Also, we have $\phi(n) = n \sum_{d|n} \mu(d)/d$, and hence

$$\begin{aligned} \sum_{1 \leq n \leq x} \frac{\phi(n)}{n^2} &= \sum_{1 \leq n \leq x} \frac{1}{n} \sum_{d|n} \mu(d)/d = \sum_{1 \leq d \leq x} \sum_{1 \leq m \leq x/d} \frac{\mu(d)}{md^2} \\ &= \sum_{1 \leq m \leq x} \frac{1}{m} \sum_{1 \leq d \leq x/m} \frac{\mu(d)}{d^2} = \sum_{1 \leq m \leq x} \frac{1}{m} \left(\frac{6}{\pi^2} + O(m/x) \right) \\ &= \frac{6}{\pi^2} \log x + O(1). \end{aligned}$$

2. (a) By multiplicativity, one has

$$\frac{\mu^2(d)}{d^2} = \prod_{p|d} p^{-2},$$

when d is squarefree, and $\mu^2(d)/d^2 = 0$ otherwise, and hence

$$\prod_p (1 + 1/p^2) = \sum_{\substack{d=1 \\ d \text{ squarefree}}}^{\infty} \prod_{p|d} p^{-2} = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2}.$$

(b) Then

$$\sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} = \frac{\prod_p (1 - 1/p^2)^{-1}}{\prod_p (1 - 1/p^4)^{-1}} = \frac{\zeta(2)}{\zeta(4)}.$$

Thus

$$\sum_{1 \leq d \leq x} \frac{\mu^2(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} + O\left(\sum_{d>x} \frac{1}{d^2} \right) = \frac{\pi^2/6}{\pi^4/90} + O(1/x) = \frac{15}{\pi^2} + O(1/x).$$

3. Define

$$\beta(n) = \sum_{\substack{a=1 \\ (a,n)=1}}^n \log(a/n).$$

Then, applying the formula that we are asked to recall in the question, one obtains

$$\begin{aligned} \beta(n) &= \sum_{d|n} \sum_{\substack{a=1 \\ d|a}}^n \mu(d) \log(a/n) = \sum_{d|n} \mu(d) \sum_{b=1}^{n/d} \log\left(\frac{b}{n/d}\right) \\ &= \sum_{d|n} \mu(n/d) \sum_{b=1}^d \log(b/d) = \sum_{d|n} \mu(n/d) \log(d!/d^d). \end{aligned}$$

Consequently, one finds that

$$\prod_{\substack{a=1 \\ (a,n)=1}}^n a = n^{\phi(n)} e^{\beta(n)} = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}.$$

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