SOLUTIONS TO HOMEWORK 9

1. (a) The function $\mu(n)$ is multiplicative, and hence $\mu^2(n)$ is also multiplicative. Then it suffices to examine prime powers, where we find that for each prime p and non-negative integer h, one has

$$\sum_{d|p^h} \mu^2(d) = \sum_{l=0}^{h} \mu^2(p^l) = \begin{cases} 1, & \text{when } h = 0, \\ 1 + \mu(p)^2 = 2, & \text{when } h \geqslant 1. \end{cases}$$

Thus, by applying multiplicativity, we see that when $n = \prod_{p^h \| n} p^h$, one has $\sum_{d|n} \mu^2(d) = \prod_{p|n} 2 = 2^{\omega(n)}$, as required.

(b) Since $\tau(n)$ is also multiplicative, we may proceed in like manner. Here we note that $\tau(p^l) = l + 1$, and hence

$$\sum_{d|p^h} \mu(d)\tau(d) = \sum_{l=0}^{h} \mu(p^l)\tau(p^l) = \begin{cases} 1, & \text{when } h = 0, \\ 1 - 2 = -1, & \text{when } h \geqslant 1. \end{cases}$$

Thus, by applying multiplicativity, we see that when $n = \prod_{p^h \| n} p^h$, one has $\sum_{d|n} \mu(d)\tau(d) = \prod_{p|n}(-1) = (-1)^{\omega(n)}$, as required.

2. (a) The sum of the first n positive integers is $n(n+1)/2$, so

$$\left(\sum_{a=1}^{n} a \right)^2 = (n(n+1)/2)^2 = \tfrac{1}{4} n^2 (n+1)^2.$$

Meanwhile, whenever

$$\sum_{a=1}^{n} a^3 = \tfrac{1}{4} n^2 (n+1)^2,$$

then one has

$$\sum_{a=1}^{n+1} a^3 = (n+1)^3 + \tfrac{1}{4} n^2 (n+1)^2 = \tfrac{1}{4}(n+1)^2(4(n+1) + n^2) = \tfrac{1}{4}(n+1)^2(n+2)^2.$$

Since $\sum_{a=1}^{1} a^3 = 1 = \frac{1}{4} 1^2 (1+1)^2$, we conclude by induction that

$$\sum_{a=1}^{n} a^3 = \tfrac{1}{4} n^2 (n+1)^2 = \left(\sum_{a=1}^{n} a \right)^2.$$

(b) For each prime power p^h, we have

$$\sum_{a=0}^{h} \tau(p^a) = \sum_{a=0}^{h} (a+1) = \tfrac{1}{2}(h+1)(h+2),$$

and

$$\sum_{a=0}^{h} \tau(p^a)^3 = \sum_{a=0}^{h} (a+1)^3 = \tfrac{1}{4}(h+1)^2(h+2)^2.$$

Thus, whenever n is a prime power, one has

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d) \right)^2,$$

and the desired conclusion follows by multiplicativity.

3. Let f be an arithmetic function.

(a) When a and n are positive integers, one has

$$\sum_{d|(a,n)} \mu(d) = \nu((a,n)) = \begin{cases} 1, & \text{when } (a,n) = 1, \\ 0, & \text{when } (a,n) > 1. \end{cases}$$

(b) Thus

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} f(a) = \sum_{1 \leqslant a \leqslant n} \sum_{d|(a,n)} \mu(d) f(a) = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} f(a).$$

(c) First taking $f(a) = 1$, we find that

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} 1 = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} 1 = \sum_{d|n} \mu(d) n/d = \varphi(n).$$

Next, taking $f(a) = a$, we obtain

$$\sum_{\substack{1 \leqslant a \leqslant n \\ (a,n)=1}} a = \sum_{d|n} \mu(d) \sum_{\substack{1 \leqslant a \leqslant n \\ d|a}} a = \sum_{d|n} \mu(d) d \cdot \tfrac{1}{2}(n/d)(n/d+1)$$

$$= \tfrac{1}{2} n \sum_{d|n} \mu(d) n/d + \tfrac{1}{2} n \sum_{d|n} \mu(d) = \tfrac{1}{2} n \varphi(n).$$

4. (a) Suppose that n and m are coprime with $n = \prod_{p^h \| n} p^h$ and $m = \prod_{\pi^h | m} \pi^h$, say, with p and π denoting prime numbers. Since $(n,m) = 1$, the primes p and π occurring in these products are distinct, and thus

$$s(nm) = \prod_{p|nm} p = \left(\prod_{p|n} p \right) \left(\prod_{\pi|m} \pi \right) = s(n)s(m).$$

Moreover, one has $s(1) = 1$, and so we conclude that $s(n)$ is a multiplicative function of n.

(b) By Möbius inversion, the arithmetic function $f(n)$ defined by putting

$$f(n) = \sum_{d|n} \mu(d) s(n/d)$$

satisfies the property that $s(n) = \sum_{d|n} f(d)$. But $\mu(n)$ and $s(n)$ are both multiplicative functions, and thus f is also a multiplicative function. We have $f(1) = 1$, and when p is prime and $h \geqslant 1$,

$$f(p^h) = \sum_{a=0}^{h} \mu(p^a)s(p^{h-a}) = s(p^h) - s(p^{h-1}) = \begin{cases} p-1, & \text{when } h = 1, \\ p - p = 0, & \text{when } h \geqslant 2. \end{cases}$$

Thus, in all cases one has $f(p^h) = \mu^2(p^h)\varphi(p^h)$, and by multiplicativity we conclude that $f(n) = \mu^2(n)\varphi(n)$.

5. (a) Suppose that $a(n)$ and $b(n)$ are multiplicative. Then whenever $m, n \in \mathbb{N}$ satisfy $(m, n) = 1$, we have $a(mn) = a(m)a(n)$ and $b(mn) = b(m)b(n)$, whence

$$c(mn) = \sum_{d|mn} a(nm/d)b(d) = \sum_{e|n}\sum_{f|m} a\left(\frac{n}{e}\frac{m}{f}\right) b(ef).$$

Since the values of e and f in the latter summation are necessarily coprime, we find that

$$\begin{aligned} c(mn) &= \sum_{e|n}\sum_{f|m} a(n/e)a(m/f)b(e)b(f) \\ &= \left(\sum_{e|n} a(n/e)b(e)\right)\left(\sum_{f|m} a(m/f)b(f)\right) = c(m)c(n). \end{aligned}$$

Thus $c(n)$ is indeed a multiplicative function.

(b) Let $a(n) = \phi(n)$ and $b(n) = \tau(n)$. Then for each prime power p^h one has

$$\begin{aligned} \sum_{j=0}^{h} \phi(p^{h-j})\tau(p^j) &= \sum_{j=0}^{h-1}(p^{h-j} - p^{h-j-1})(j+1) + \phi(p^0)\tau(p^h) \\ &= p^h + p^{h-1} + \cdots + p - h + h + 1 = \sum_{d|p^h} d = \sigma(p^h), \end{aligned}$$

and so

$$\sigma(p^h) = \sum_{d|p^h} \phi(p^h/d)\tau(d).$$

Thus it follows from multiplicativity that $\sigma(n) = \sum_{d|n} \phi(n/d)\tau(d)$ for $n \in \mathbb{N}$.