MA59800ANT ANALYTIC THEORY OF FUNCTION FIELDS. PROBLEMS 4

TO BE HANDED IN BY 6PM MONDAY 28TH OCTOBER 2024

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

Throughout, we take p to be a prime number, put $q = p^h$, and denote $\mathbb{F}_q[t]$ by $\mathbb{A}$.

A1. Consider the polynomials $u \in \mathbb{A}$ of the shape

$$u(t) = t^n + u_{n-1}t^{n-1} + \ldots + u_k t^k + \ldots + u_1 t + u_0,$$

with $u_i \in \mathbb{F}_q$.

(a) Show that as $n \to \infty$, there are infinitely many irreducible polynomials of this shape with $u_0, u_1, \ldots, u_{k-1}$ all fixed to be specified values, provided that $u_0 \neq 0$. Compute the Dirichlet density of this set of polynomials.

(b) Show that, as $n \to \infty$, there are infinitely many irreducible polynomials of this shape with $u_{n-1}, \ldots, u_{n-k}$ all fixed to be specified values. Compute the Dirichlet density of this set of polynomials.

A2. (a) Show that, whenever $c \in \mathbb{F}_q$ and $d \in \mathbb{F}_q \setminus \{0\}$, there are infinitely many monic irreducible polynomials $\pi(t) \in \mathbb{A}$ having the property that $\pi(c) = d$.

(b) Show that, whenever c_1 and c_2 are distinct elements of $\mathbb{F}_q$, and $d_1, d_2 \in \mathbb{F}_q \setminus \{0\}$, then there are infinitely many monic irreducible polynomials $\pi(t) \in \mathbb{A}$ having the property that $\pi(c_1) = d_1$ and $\pi(c_2) = d_2$.

B3. Assume the Riemann Hypothesis for Dirichlet L-functions over $\mathbb{F}_q[t]$. Let $\Lambda(u)$ denote the von Mangoldt function in $\mathbb{A} = \mathbb{F}_q[t]$. Suppose that $a, m \in \mathbb{F}_q[t]$ satisfy $(a, m) = 1$ and $\deg(m) \geqslant 1$. Obtain asymptotic formulae for the following quantities as $n \to \infty$:

$$\sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u) = n \\ u \equiv a \pmod{m}}} \Lambda(u) \quad \text{and} \quad \sum_{\substack{u \in \mathbb{F}_q[t]^+ \\ \deg(u) = n \\ u \equiv a \pmod{m}}} \frac{\Lambda(u)}{|u|}.$$

B4. Recall that the forward difference operator Δ_1 is defined for any $\mathbb{K}_\infty$-valued function ψ of the variable x via the relation

$$\Delta_1(\psi(x); h) = \psi(x + h) - \psi(x).$$

The j-fold forward difference operator is then defined for $j \geqslant 2$ inductively via the relation

$$\Delta_j(\psi(x); h_1, \ldots, h_j) = \Delta_1(\Delta_{j-1}(\psi(x); h_1, \ldots, h_{j-1}); h_j).$$

Show that for $1 \leqslant j \leqslant k$, one has

$$\Delta_j(x^k; h_1, \dots, h_j) = \sum_{i_0,\dots,i_j} \frac{k!}{i_0! \dots i_j!} x^{i_0} h_1^{i_1} \dots h_j^{i_j},$$

where the summation is over $i_0 \geqslant 0, i_1 \geqslant 1, \dots, i_j \geqslant 1$ and $i_0 + \dots + i_j = k$. Hence deduce that

$$\Delta_j(x^k; h_1, \dots, h_j) = h_1 \dots h_j p_j(x; h_1, \dots, h_j),$$

where p_j is a polynomial in x of degree $k-j$ with leading coefficient $k!/(k-j)!$.

B5. Let H and X be positive integers. Suppose that $k \geqslant 2$ and $\alpha \in \mathbb{K}_\infty$, and define

$$T(\alpha) = \sum_{0 \leqslant \deg(h) < H} \Bigl| \sum_{\deg(x) < X} e(h\alpha x^k) \Bigr|.$$

(a) Show that

$$T(\alpha)^{2^{k-1}} \leqslant (q^H)^{2^{k-1}-1} \sum_{0 \leqslant \deg(h) < H} \Bigl| \sum_{\deg(x) < X} e(h\alpha x^k) \Bigr|^{2^{k-1}}.$$

(b) Suppose that $a, g \in \mathbb{F}_q[t]$ satisfy $(a, g) = 1$, g monic, and $|\alpha - a/q| < 1/|g|^2$. Show that, whenever $\mathrm{char}(\mathbb{F}_q) > k$, one has

$$T(\alpha) \ll (q^{H+X})^{1+\varepsilon} \left(|g|^{-1} + q^{-X} + |g|(q^{H+kX})^{-1} \right)^{2^{1-k}}.$$

C6. Suppose that $\mathrm{char}(\mathbb{F}_q) \neq 2$, and put $\alpha = \sqrt{1+t^2} \in \mathbb{K}_\infty$.

(a) Show that the continued fraction expansion of α is $\alpha = [t; 2t, 2t, \dots]$.

(b) Suppose that the convergents to the continued fraction expansion of α are a_n/g_n ($n \in \mathbb{N}$). By examining $a_n^2 - (1+t^2)g_n^2$, find an infinite sequence of polynomials $(x_n, y_n)_{n=1}^\infty$ having the property that $x_n^2 - (t^2+1)y_n^2 = 1$.

(c) Find an explicit constant $c > 0$ having the property that, whenever a and g are elements of $\mathbb{F}_q[t]$ with g monic and $(a, g) = 1$, then $|\sqrt{1+t^2} - a/g| > c/|g|^2$.

C7. Suppose that $\theta \in \mathbb{K}_\infty$ is irrational.

(a) Show that the sequence of squares of fractional parts of θx, namely

$$\left(\{\theta x\}^2 \right)_{x \in \mathbb{F}_q[t]},$$

is *not* equidistributed in $\mathbb{T}$.

(b) Suppose that $\mathrm{char}(\mathbb{F}_q)$ is odd. Show that there exists an irrational element $\theta \in \mathbb{K}_\infty$ having the property that

$$\left(x \lfloor x\sqrt{1+t^2} \rfloor \theta \right)_{x \in \mathbb{F}_q[t]},$$

is *not* equidistributed in $\mathbb{T}$.