MA59800ANT ANALYTIC THEORY OF FUNCTION FIELDS. PROBLEMS 5

TO BE HANDED IN BY 6PM FRIDAY 8TH NOVEMBER 2024

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

Throughout, we take p to be a prime number, put $q = p^h$, and denote $\mathbb{F}_q[t]$ by $\mathbb{A}$.

A1. Suppose throughout that $k \geqslant 2$.
(a) Prove that

$$\int_{\mathbb{T}} \Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr|^2 \, \mathrm{d}\alpha \geqslant q^{X+1}.$$

(b) Apply Hölder's inequality to deduce that for each real number s with $s \geqslant 1$, one has

$$\int_{\mathbb{T}} \Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr|^{2s} \, \mathrm{d}\alpha \geqslant (q^{X+1})^s.$$

A2. Suppose throughout that $k \geqslant 2$.
(a) Show that when $|\alpha| < q^{-1}(q^X)^{-k}$, one has

$$\Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr| = q^{X+1}.$$

(b) Prove that when s and k are positive integers, one has

$$\int_{\mathbb{T}} \Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr|^{2s} \, \mathrm{d}\alpha \geqslant q^{k-1}(q^{X+1})^{2s-k}.$$

B3. Suppose throughout that $k \geqslant 2$.
(a) Show that when $s \geqslant 1$, one has

$$\int_{\mathbb{T}} \Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr|^{2s} \, \mathrm{d}\alpha \geqslant (q^X)^s + (q^X)^{2s-k}.$$

(b) Suppose that $0 < s < 1$ and $p > k$. By applying Hua's lemma, show that for each $\varepsilon > 0$, one has

$$\int_{\mathbb{T}} \Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr|^{2s} \, \mathrm{d}\alpha \gg (q^X)^{s-\varepsilon}.$$

B4. Suppose that $k \in \mathbb{N}$, that X is a large real number, and that Q is a real number with $1 \leqslant Q < \frac{1}{2}kX$. Denote by $\mathfrak{M}(Q)$ the union of the arcs

$$\mathfrak{M}(g,a) = \{\alpha \in \mathbb{T} : |g\alpha - a| < q^Q (q^X)^{-k}\},$$

with $a, g \in \mathbb{F}_q[t]$, g monic, $0 \leqslant |a| < |g| \leqslant q^Q$ and $(a,g) = 1$, and put $\mathfrak{m}(Q) = \mathbb{T} \setminus \mathfrak{M}(Q)$.

(a) Suppose that $\alpha \in \mathfrak{M}(Q)$ and $r \in \mathbb{F}_q[t] \setminus \{0\}$. Show that $r\alpha \in \mathfrak{M}(Q + \deg(r)) + \mathbb{A}$.

(ii) Suppose that $r\alpha \in \mathfrak{M}(Q) + \mathbb{A}$ with $r \in \mathbb{F}_q[t] \setminus \{0\}$. Show that $\alpha \in \mathfrak{M}(Q + \deg(r)) + \mathbb{A}$.

(iii) Suppose that $\alpha \in \mathfrak{m}(Q)$ and $r \in \mathbb{F}_q[t] \setminus \{0\}$. Show that $r\alpha \in \mathfrak{m}(Q - \deg(r)) + \mathbb{A}$.

(iv) Suppose that $r\alpha \in \mathfrak{m}(Q) + \mathbb{A}$ with $r \in \mathbb{F}_q[t] \setminus \{0\}$. Show that $\alpha \in \mathfrak{m}(Q - \deg(r)) + \mathbb{A}$.

B5. Suppose throughout that $k \geqslant 2$ and $p > k$. Write

$$S(g,a) = \sum_{|r|<|g|} e(ar^k/g).$$

(a) Suppose that $a, g \in \mathbb{F}_q[t]$, g monic, $0 \leqslant |a| < |g|$ and $(a,g) = 1$. Show that

$$S(g,a) \ll |g|^{1-2^{1-k}+\varepsilon}.$$

(b) Show that when s is an integer with $s > 2^k$, one has

$$\sum_{\substack{|a|<|g| \\ (a,g)=1}} |S(g,a)|^s \ll |g|^{s-1-2^{-k}}.$$

C6. (a) When $0 \leqslant j \leqslant k$, write $\tau = \tau(k,j)$ for the non-negative integer satisfying $p^\tau \| \binom{k}{j}$. Show that

$$\tau = \sum_{l=1}^{\infty} \left(\left\{ \frac{j}{p^l} \right\} + \left\{ \frac{k-j}{p^l} \right\} - \left\{ \frac{k}{p^l} \right\} \right),$$

where $\{\theta\} = \theta - \lfloor \theta \rfloor$.

(b) Let the base p expansion of k be $k = a_n p^n + \ldots + a_1 p + a_0$, with $0 \leqslant a_i < p$ for each i. Show that $\binom{k}{j}$ is coprime to p if and only if j has the shape $j = b_n p^n + \ldots + b_1 p + b_0$, with $0 \leqslant b_i \leqslant a_i$ for each i.

(c) Apply induction to show that $\Delta_j(x^k; \mathbf{h})$ is not indentically 0 as a polynomial in x whenever $0 \leqslant j \leqslant a_0 + \ldots + a_n$.

C7. Let the base p expansion of k be $k = a_n p^n + \ldots + a_1 p + a_0$, with $0 \leqslant a_i < p$ for each i. Prove that whenever $1 \leqslant j \leqslant a_0 + \ldots + a_n$, then for each positive number ε, one has

$$\int_{\mathbb{T}} \Bigl| \sum_{|x| \leqslant q^X} e(\alpha x^k) \Bigr|^{2^j} \mathrm{d}\alpha \ll (q^X)^{2^j - j + \varepsilon}.$$