MA59800ANT ANALYTIC THEORY OF FUNCTION FIELDS. PROBLEMS 6

TO BE HANDED IN BY 6PM WEDNESDAY 4TH DECEMBER 2024

Key: A-questions are short questions testing basic skill sets; B-questions integrate essential methods of the course; C-questions are more challenging for enthusiasts, with hints available on request.

Throughout, we take p to be a prime number, put $q = p^h$, and denote $\mathbb{F}_q[t]$ by $\mathbb{A}$.

A1. Suppose that $N \in \mathbb{K}_\infty$. Use the definition of the function $e(\alpha)$ to prove that when $n \geqslant 0$, one has

$$\int_{|\alpha|<q^{-n}} e(N\alpha)\,\mathrm{d}\alpha = \begin{cases} q^{-n}, & \text{when } |N| < q^n, \\ 0, & \text{otherwise.} \end{cases}$$

A2. Suppose that $(k, q-1) = 1$. Prove that for each $m \in \mathbb{F}_q[t]$, one has

$$J_\infty(m) = \begin{cases} q^{s-1}, & \text{when } k|\deg(m) \text{ and } m \text{ is not exceptional,} \\ q^{s-1} - 1, & \text{otherwise.} \end{cases}$$

B3. Suppose that $k \in \mathbb{N}$ and $\mathrm{char}(\mathbb{F}_q) \nmid k$. Let π be a monic irreducible polynomial.
(i) Show that

$$\sum_{0\leqslant|u_1|<|\pi|}\sum_{0\leqslant|u_2|<|\pi|} e\left(\frac{a(u_1+u_2\pi)^k}{\pi^2}\right) = \sum_{0\leqslant|u_1|<|\pi|} e\left(\frac{au_1^k}{\pi^2}\right)\sum_{0\leqslant|u_2|<|\pi|} e\left(\frac{kau_1^{k-1}u_2}{\pi}\right).$$

(ii) Deduce that whenever $(a,\pi) = 1$, one has $S(\pi^2, a) = |\pi|$.

B4. (a) Let $k \geqslant 2$. Suppose that π is monic and irreducible in $\mathbb{A}$, and moreover one has $|\pi| > (k-1)^2$. Prove (without assuming that $\mathrm{char}(\mathbb{F}_q) > k$) that every polynomial m in $\mathbb{A}$ is congruent to a sum of k-th powers modulo π. [Hint: study the proof of Lemma 19.4].
(b) Deduce that when $|\pi| > (k-1)^2$ and $s \geqslant k+1$, then for each $m \in \mathbb{A}$, there exist $w_1, \ldots, w_s \in \mathbb{A}$ such that $\pi \nmid w_1$ and $w_1^k + \ldots + w_s^k \equiv m \pmod{\pi}$.

B5. When $r \in \mathbb{N}$, let

$$I_r^*(P) = \int_{|\beta|<(q^{P+1})^{1-k}} |f(\beta;P)|^{2r}\,\mathrm{d}\beta.$$

(a) Show that when $\mathrm{char}(\mathbb{F}_q) \nmid k$ and $r \geqslant k+1$, there is a constant $J_\infty^* = J_\infty^*(q,r,k)$ for which

$$I_r^*(P) = J_\infty^*(q^P)^{2r-k} + O((q^P)^{2r-k-1/k}).$$

(b) Show that when $\mathrm{char}(\mathbb{F}_q) \nmid k$ and $r \geqslant k+1$, one has $J_\infty^* \geqslant q^r - 1$.

C6. When $r \in \mathbb{N}$, let

$$\mathfrak{S} = \sum_{g \in \mathbb{A}^+} \sum_{\substack{0 \leqslant |a| < |g| \\ (a,g)=1}} |g|^{-2r} |S(g,a)|^{2r}.$$

(a) Show that when $\mathrm{char}(\mathbb{F}_q) > k$ and $r \geqslant 2^{k-1} + 1$, the infinite series $\mathfrak{S}$ converges absolutely.

(b) Show that, under the same conditions, one has $\mathfrak{S} = \prod_\pi T^*(\pi)$, where

$$T^*(\pi) = \sum_{l=0}^{\infty} \sum_{\substack{0 \leqslant |a| < |\pi^l| \\ (a,\pi)=1}} |\pi^l|^{-2r} |S(\pi^l, a)|^{2r}.$$

(c) Show that, under the same conditions, one has $1 \leqslant \mathfrak{S} \ll 1$.

C7. Obtain an asymptotic formula for

$$\int_{\mathbb{T}} |f(\alpha; P)|^{2r} \, \mathrm{d}\alpha,$$

valid whenever $\mathrm{char}(\mathbb{F}_q) > k$ and $r \geqslant 2^{k-1} + 1$.