GALOIS THEORY: HOMEWORK 10

Due 6pm Wednesday 27th March 2024

1. Let $f \in K[t] \setminus K$, and let $L : K$ be a splitting field extension for f. Assume that $K \subseteq L$.
   (a) Show that when f has a repeated root over L, then there exists $\alpha \in L$ for which $f(\alpha) = 0 = (Df)(\alpha)$.
   (b) Show that when $\alpha \in L$ satisfies $f(\alpha) = 0 = (Df)(\alpha)$, then there exists $g \in K[t]$ having the property that $\deg g \geq 1$ and g divides both f and Df.
   (c) Show that when $g \in K[t] \setminus K$ divides both f and Df, then f has a repeated root over L.
2. Suppose that $\mathrm{char}(K) = p > 0$ and f is irreducible over $K[t]$.
   (a) Show that there is an irreducible and separable polynomial $g \in K[t]$ and a non-negative integer n with the property that $f(t) = g(t^{p^n})$.
   (b) Let $L : K$ be a splitting field extension for f. Show that there exists a non-negative integer n with the property that every root of f in L has multiplicity p^n.
3. Revise for the second mid-term!