GALOIS THEORY: HOMEWORK 12

Due 6pm Wednesday 10th April 2024

1. Let $L : K$ be a finite Galois extension with Galois group G. For any $\alpha \in L$, define the polynomial $f_\alpha(t) = \prod_{\sigma \in G}(t - \sigma(\alpha))$.
   (a) Show that $f_\alpha \in K[t]$.
   (b) Prove that if $\sigma(\alpha) \neq \tau(\alpha)$ whenever $\sigma, \tau \in G$ satisfy $\sigma \neq \tau$, then $f_\alpha = m_\alpha(K)$.
2. Use question 1 to calculate the minimal polynomial of $2\sqrt{-3} - \sqrt{2}$ over $\mathbb{Q}$.
3. Let f denote the polynomial $t^3 + t + 1$.
   (a) Write down a splitting field extension for f over $\mathbb{F}_2$.
   (b) What is $\mathrm{Gal}_{\mathbb{F}_2}(f)$? Justify your answer, and determine all subfields of the splitting field that you wrote down in part (a).
4. Let f denote the polynomial $t^4 + t^3 + t^2 + t + 1$.
   (a) Write down a splitting field extension for f over $\mathbb{Q}$.
   (b) Show that $\mathrm{Gal}_{\mathbb{Q}}(f) \cong C_4$, where C_4 is the cyclic group of order 4.
5. Use the Galois correspondence to determine all subfields of the splitting field that you wrote down in part (a) of question 4. Draw the lattice of subfields and corresponding lattice of subgroups of C_4.

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