GALOIS THEORY: HOMEWORK 13

Due 6pm Wednesday 17th April 2024

1. Let f denote the polynomial $t^3 - 7$.
   (a) Write down a splitting field extension for f over $\mathbb{Q}$.
   (b) Show that $\mathrm{Gal}_{\mathbb{Q}}(f) \cong S_3$.
2. Use the Galois correspondence to determine all subfields of the splitting field that you wrote down in part (a) of question 1. Draw the lattice of subfields and corresponding lattice of subgroups of S_3.
3. Suppose that L is a finite field having p^n elements, where p is a prime number. Recall that $\mathrm{Gal}(L : \mathbb{F}_p) = \langle \varphi \rangle$, where φ denotes the Frobenius mapping.
   (a) Show that whenever K is a subfield of L, then $|K| = p^d$ for some divisor d of n.
   (b) Show that for each divisor d of n, there is a unique subfield K of L with $|K| = p^d$.
4. Let $L : K$ be a finite Galois extension with Galois group G.
   (a) For any $\alpha \in L$, define the *norm* of α by $\mathrm{N}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha)$. Show that $\mathrm{N}(\alpha) \in K$.
   (b) For any $\alpha \in L$, define the *trace* of α by $\mathrm{Tr}(\alpha) = \sum_{\sigma \in G} \sigma(\alpha)$. Show that $\mathrm{Tr}(\alpha) \in K$.
5. Let p be a prime number, and n a natural number, and denote by $\mathbb{F}_q$ the finite field of $q = p^n$ elements with prime field $\mathbb{F}_p$. Let ϕ denote the Frobenius monomorphism from $\mathbb{F}_q$ into $\mathbb{F}_q$. Recall that $\mathrm{Gal}(\mathbb{F}_q : \mathbb{F}_p) = \langle \phi \rangle$.
   (a) Defining the trace of $\alpha \in \mathbb{F}_q$ as in question 4(b) above, show that there exists an element $\alpha \in \mathbb{F}_q$ having non-zero trace.
   (b) Defining the norm of $\alpha \in \mathbb{F}_q$ as in question 4(a) above, show that there exists a non-zero element $\alpha \in \mathbb{F}_q^\times$ having norm different from 1.