GALOIS THEORY: HOMEWORK 14

Not for assessment – solutions will be provided

1. (a) Show that $f = t^3 - 3t + 1$ is irreducible over $\mathbb{Q}$.
   (b) Show that whenever α is a root of f in a splitting field extension of $\mathbb{Q}$, then $\beta = \alpha^2 - 2$ is also a root of f.
   (c) Let L be a splitting field for f over $\mathbb{Q}$. Use your answer to part (b) to show that $[L : \mathbb{Q}] = 3$, and conclude that the Galois group of f is isomorphic to $A_3 \cong C_3$.
   (d) Show that there is no $\gamma \in L$ such that $\gamma \notin \mathbb{Q}$ and $\gamma^3 \in \mathbb{Q}$, and conclude that $L : \mathbb{Q}$ is not a radical extension.
   (e) By Cardano's formula, the equation $f = 0$ is soluble by radicals. How do you reconcile this observation with your answer to part (d)?
2. Is the polynomial $t^5 - 4t^4 + 2$ soluble by radicals over $\mathbb{Q}$?
3. Is the polynomial $t^6 - 4t^2 + 2$ soluble by radicals over $\mathbb{Q}$?
4. Let n be a positive integer and K a field with characteristic not dividing n. Let $L = K(\zeta)$, where ζ is a primitive nth root of unity.
   (a) Show that $\mathrm{Gal}(L : K)$ is isomorphic to a subgroup of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$.
   (b) Show that if n is prime and $K = \mathbb{Q}$ then either $L = K$ or $\mathrm{Gal}(L : K) \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
5. Let n be a positive integer. By Dirichlet's theorem, there exists a prime number p with $p \equiv 1 \pmod{n}$.
   (a) Let $L = \mathbb{Q}(e^{2\pi i/p})$. Show that $\mathrm{Gal}(L : \mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$.
   (b) Show that $\mathbb{Q}(e^{2\pi i/p})$ contains a subfield M with the property that $\mathrm{Gal}(M : \mathbb{Q}) \cong C_n$.