GALOIS THEORY: HOMEWORK 2

Due 6pm Wednesday 24th January 2024

1. Let $L : K$ be a field extension, and suppose that $\theta \in L$ satisfies the property that $[K(\theta) : K] = p$, where p is a prime number. Let
$$\alpha = c_0 + c_1\theta + \ldots + c_{p-1}\theta^{p-1},$$
for some $c_0, \ldots, c_{p-1} \in K$, and suppose that $\alpha \notin K$. By considering $[K(\alpha) : K]$, show that $K(\alpha) = K(\theta)$.
2. Let $L : K$ be a field extension with $K \subseteq L$. Let $A \subseteq L$, and let
$$\mathcal{C} = \{C \subseteq A : C \text{ is a finite set}\}.$$
Show that $K(A) = \cup_{C \in \mathcal{C}} K(C)$, and further that when $[K(C) : K] < \infty$ for all $C \in \mathcal{C}$, then $K(A) : K$ is an algebraic extension.
3. Let $L : K$ be a field extension, and suppose that $\gamma \in L$ satisfies the property that $\deg m_\gamma(K) = 5$. Suppose that $h \in K[t]$ is a non-zero cubic polynomial. By noting that γ is a root of the cubic polynomial $g(t) = h(t) - h(\gamma) \in K(h(\gamma))[t]$, show that $[K(h(\gamma)) : K] = 5$.
4. Calculate the minimal polynomial of $\sqrt[5]{7 + \sqrt[3]{21}}$ over $\mathbb{Q}$, and hence determine the degree of the field extension $\mathbb{Q}(\sqrt[5]{7 + \sqrt[3]{21}}) : \mathbb{Q}$.
5. Let $\mathbb{Q}(\alpha) : \mathbb{Q}$ be a simple field extension with the property that the minimal polynomial of α is $t^3 + 2t - 2$. Calculate the minimal polynomials of $\alpha - 1$ and $\alpha^2 + 1$ over $\mathbb{Q}$, and express the multiplicative inverses of these elements in $\mathbb{Q}(\alpha)$ in the form $c_0 + c_1\alpha + c_2\alpha^2$ for suitable rational numbers c_0, c_1, c_2.