GALOIS THEORY: HOMEWORK 3

Due 6pm Wednesday 31st January 2024

1. (a) Show that when p is a prime number, then for every positive integer n the polynomial $X^n - p$ is irreducible over $\mathbb{Q}[X]$.
(b) By making the substitution $y = X - 1$, or otherwise, show that when p is a prime number, the polynomial $X^{p-1} + X^{p-2} + \cdots + X + 1$ is irreducible over $\mathbb{Q}$.
2. (a) Show that the polynomial $\phi = t^3 - t + 1$ is irreducible over the ring $\mathbb{I} = \mathbb{F}_3[t]$.
(b) Let $\mathbb{K} = \mathbb{F}_3(t)$. Show that the polynomial $X^{2024} + \phi X^2 + \phi$ is irreducible over $\mathbb{K}[X]$.
3. Let $L : K$ be a field extension. Suppose that $\alpha \in L$ is algebraic over K and $\beta \in L$ is transcendental over K. Suppose also that $\alpha \notin K$. Show that $K(\alpha, \beta) : K$ is not a simple field extension.
4. (a) Show that the polynomial $f(t) = t^7 - 7t^5 + 14t^3 - 7t - 2$ factorises over $\mathbb{Q}[t]$ in the form $f = g_1 g_3^2$, where $g_1, g_3 \in \mathbb{Z}[t]$ have the property that g_1 is linear, and g_3 is cubic and irreducible.
(b) Using the identity
$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta,$$
together with the conclusion of part (a), show that the angle $2\pi/7$ is not constructible by ruler and compass. Hence deduce that the regular heptagon is not constructible by ruler and compass.
5. Assume (as has in fact been proved) that $\pi = 3.14159\ldots$ is transcendental over $\mathbb{Q}$.
(a) Show that one cannot "square the circle" – that is, prove that $\sqrt{\pi}$ is not constructible by ruler and compass.
(b) Suppose that a generous benefactor has given you the points $(0,0)$, $(0,1)$ and $(0,\pi)$ in the plane. Can you now construct $\pi^{1/5}$ by ruler and compass from these three points? Explain your answer.