GALOIS THEORY: HOMEWORK 4

Due 6pm Wednesday 7th February 2024

1. (a) By considering the substitution $t = x + 1$ and applying Eisenstein's criterion, show that the polnomial $t^6 + t^3 + 1$ is irreducible over $\mathbb{Q}[t]$.
   (b) Suppose, if possible, that $[\mathbb{Q}(\cos(2\pi/9), \sin(2\pi/9)) : \mathbb{Q}] = 2^r$, for some non-negative integer r. Prove that the 9-th root of unity $\omega = \cos(2\pi/9) + i\sin(2\pi/9)$ satisfies the property that $[\mathbb{Q}(\omega) : \mathbb{Q}]$ divides 2^{r+1}.
   (c) By considering the factorisation of $t^9 - 1$ over $\mathbb{Q}[t]$, prove that $[\mathbb{Q}(\omega) : \mathbb{Q}] = 6$. Hence deduce that the angle $2\pi/9$ is not constructible by ruler and compass, whence the regular nonagon cannot be constructed by ruler and compass.

2. (a) Suppose that $P_0, P_1, \ldots, P_n$ are points in $\mathbb{R}^2$ whose coordinates lie in a field extension K of $\mathbb{Q}$. Let $P = (x, y)$ be a point of intersection of two ellipses with equations defined over K. Explain why $[K(x, y) : K] \leq 4$.
   (b) Let $P_0 = (0, 0)$ and $P_1 = (1, 0)$, and suppose that $P_2, P_3, \ldots$ are constructed successively by simple cord-and-nail constructions (as discussed in Definition 13 of section 2.3 from the notes). Let j be a positive integer, write $P_j = (x_j, y_j)$, and put $L_j = \mathbb{Q}(x_j, y_j)$. Explain why, for some non-negative integers r and s, one has $[L_j : \mathbb{Q}] = 2^r 3^s$.

3. (a) Prove that the polynomial $t^5 - 2$ is irreducible over $\mathbb{Q}[t]$.
   (b) Prove that $2^{1/5}$ does not lie in any field extension L of $\mathbb{Q}$ with $[L : \mathbb{Q}] = 2^r 3^s$, for any non-negative integers r and s. (This shows that $2^{1/5}$ is not simply constructible by cord-and-nail).

4. Suppose that $L : K$ is a field extension with $K \subseteq L$, and that $\tau : L \to L$ is a K-homomorphism. Suppose also that $f \in K[t]$ has the property that $\deg f \geq 1$, and additionally that $\alpha \in L$.
   (a) Show that when $f(\alpha) = 0$, then $f(\tau(\alpha)) = 0$.
   (b) Deduce that when τ is a K-automorphism of L, we have that $f(\alpha) = 0$ if and only if $f(\tau(\alpha)) = 0$.

5. Let $L : K$ be a field extension. Show that $\mathrm{Gal}(L : K)$ is a subgroup of $\mathrm{Aut}(L)$.