GALOIS THEORY: HOMEWORK 7

Due 6pm Wednesday 28th February 2024

1. Suppose that $\overline{K}$ is an algebraic closure of K, and assume that $K \subseteq \overline{K}$. Take $\alpha \in \overline{K}$ and suppose that $\sigma : K \to \overline{K}$ is a homomorphism.
   (a) Show that σ can be extended to a homomorphism $\tau : \overline{K} \to \overline{K}$.
   (b) Prove that the number of distinct roots of $m_\alpha(K)$ in $\overline{K}$ is equal to the number of distinct roots of $\sigma(m_\alpha(K))$ in $\overline{K}$.
2. Suppose that $L : K$ is an algebraic extension of fields.
   (a) Show that $\overline{L}$ is an algebraic closure of K, and hence $\overline{L} \simeq \overline{K}$.
   (b) Suppose that $K \subseteq L \subseteq \overline{L}$. Show that one may take $\overline{K} = \overline{L}$.
3. For each of the following polynomials, construct a splitting field L over $\mathbb{Q}$ and compute the degree $[L : \mathbb{Q}]$.
   (a) $t^3 - 1$
   (b) $t^7 - 1$
4. For each of the following polynomials, construct a splitting field L over $\mathbb{Q}$ and compute the degree $[L : \mathbb{Q}]$.
   (a) $t^4 + t^2 - 6$
   (b) $t^8 - 16$
5. Suppose that $L : K$ is a splitting field extension for the polynomial $f \in K[t] \setminus K$.
   (a) Prove that $[L : K] \leq (\deg f)!$.
   (b) Prove that $[L : K]$ divides $(\deg f)!$.