GALOIS THEORY: HOMEWORK 8

Due 6pm Wednesday 6th March 2024

1. Recall the splitting field L over $\mathbb{Q}$ that you constructed in question 4(b) of Problem Sheet 7 for the polynomial $t^8 - 16$. Determine the subgroup of S_4 to which $\mathrm{Gal}(L : \mathbb{Q})$ is isomorphic.
2. Suppose that K is a field and that $L : K$ is a splitting field extension for an irreducible polynomial $f \in K[t]$ of degree n. Assume that $K \subseteq L$.
   (a) Show that whenever α and β are roots of f in L, and σ is a K-automorphism of L, then $\sigma(\alpha) = \sigma(\beta)$ if and only if $\alpha = \beta$;
   (b) Show that the elements of $\mathrm{Gal}(L : K)$ act as permutations on the n roots of f, and hence deduce that $\mathrm{Gal}(L : K)$ has order dividing $n!$;
   (c) Let g be a degree m polynomial in $K[t]$, not necessarily irreducible, and let $M : K$ be a splitting field extension for g. Show that $|\mathrm{Gal}(M : K)|$ divides $m!$.
3. Suppose that $L : K$ is a normal extension, and that $K \subseteq L \subseteq \overline{K}$. Recall that since $L : K$ is algebraic, then any algebraic closure of K is an algebraic closure of L.
   (a) Show that for any K-homomorphism $\tau : L \to \overline{K}$, one has $\tau(L) = L$;
   (b) Suppose that M is a field satisfying $K \subseteq M \subseteq L$. Show that $L : M$ is a normal extension.
4. Which of the following field extensions are normal? Justify your answers.
   (a) $\mathbb{Q}(\sqrt{3}) : \mathbb{Q}$
   (b) $\mathbb{Q}(\sqrt[3]{3}) : \mathbb{Q}$
   (c) $\mathbb{Q}(\sqrt{-1}) : \mathbb{Q}$
   (d) $\mathbb{Q}(\sqrt{3}, \sqrt[3]{3}) : \mathbb{Q}$
   (e) $\mathbb{Q}(\sqrt{-1}, \sqrt{3}, \sqrt[3]{3}) : \mathbb{Q}$.
5. Let $K = \mathbb{F}_5(t)$. Find an algebraic field extension $L : K$ which is not normal, and justify your answer.