GALOIS THEORY: HOMEWORK 9

Due 6pm Wednesday 20th March 2024

1. Suppose that $E : K$ and $F : K$ are finite extensions having the property that K, E and F are contained in a field L.
   (a) Show that $EF : K$ is a finite extension;
   (b) Show that when $E : K$ and $F : K$ are both normal, then $E \cap F : K$ is a normal extension;
   (c) Show that when $E : K$ and $F : K$ are both normal, then $EF : E \cap F$ is a normal extension.
2. Suppose that $L : M$ is an algebraic extension with $M \subseteq L$. Show that when $\alpha \in L$ and $\sigma : M \to \overline{M}$ is a homomorphism, then $\sigma(m_\alpha(M))$ is separable over $\sigma(M)$ if and only if $m_\alpha(M)$ is separable over M.
3. (a) Suppose that $f \in K[t]$ is separable over K and that $L : K$ is a splitting field extension for f. Show that $L : K$ is separable.
   (b) Suppose that $L : K$ is a splitting field extension for $S \subseteq K[t]$ where each $f \in S$ is separable over K. Show that $L : K$ is a separable extension.
4. Let p be a prime number, let $\mathbb{F}_p$ denote the finite field of p elements, and let $K = \mathbb{F}_p(t)$. Suppose that $L : K$ is a field extension, and $s \in L$ is transcendental over K.
   (a) Write $J = K(s)$, and let E denote a splitting field for the polynomial $x^p - t \in J[x]$. Show that for some $\xi \in E$, one has $x^p - t = (x - \xi)^p$, and deduce that $[E : J] = p$.
   (b) Let $U : J$ be a splitting field extension for the polynomial $(x^p - t)(x^p - s)$. By considering a splitting field extension F for the polynomial $x^p - s \in E[x]$, show that $[U : J] = p^2$.
5. With the same notation as in the previous question:
   (a) Show that if $\gamma \in U$, then $\gamma^p \in J$.
   (b) What is the degree of the field extension $J(\gamma) : J$? Explain.
   (c) Deduce that $U : J$ is a finite field extension which is not simple.