GALOIS THEORY: SOLUTIONS TO HOMEWORK 11

1. Suppose that $L : M : K$ is an algebraic tower of fields. Prove that $L : K$ is separable if and only if $L : M$ and $M : K$ are both separable. [Hint: try using the Primitive Element Theorem].
Solution: We showed in Proposition 7.1 that when $L : K$ is separable, then so too is $L : M$. Meanwhile, the separability, in such circumstances, of $M : K$ is inherited from that of $L : K$. Conversely, suppose that $L : M$ and $M : K$ are both separable, and suppose that $\alpha \in L$. Then since $L : M$ is separable, one finds that α is separable over M. The polynomial $m_\alpha(M)$ has its coefficients defined in a subfield M' of M with $M' : K$ a finite separable extension. Since $m_\alpha(M') = m_\alpha(M)$ is separable, we deduce that α is separable over M'. Thus, since $M' : K$ is finite and separable, it follows from the primitive element theorem that there exists $\beta \in M'$ such that $M' = K(\beta)$, whence Theorem 7.4 implies that $M'(\alpha) : K$, or equivalently $K(\alpha, \beta) : K$, is separable. Consequently, we deduce that $\alpha \in K(\alpha, \beta)$ is separable over K. Since this conclusion holds for all $\alpha \in L$, we conclude that $L : K$ is separable.
2. Suppose that $E : K$ and $F : K$ are finite extensions with $K \subseteq E \subseteq L$ and $K \subseteq F \subseteq L$, with L a field.
   (a) Show that when $E : K$ is separable, then so too is $EF : F$.
   Solution: By the primitive element theorem, we may suppose that $E = K(\alpha)$ for some $\alpha \in E$ separable over K. Thus $EF = F(\alpha)$. Since α is separable over K, it is also separable over F, and hence it follows from Theorem 7.4 that $F(\alpha) : F$, or equivalently $EF : F$, is separable.
   (b) Show that when $E : K$ and $F : K$ are both separable, then so too are $EF : K$ and $E \cap F : K$.
   Solution: When $E : K$ and $F : K$ are both separable, then $EF : F$ is separable, and hence $EF : F : K$ is a tower of extensions with $EF : F$ and $F : K$ both separable. Then it follows from problem 1 that $EF : K$ is separable. Likewise, one has the tower $E : E \cap F : K$ of extensions with $E : K$ separable. Then it follows from problem 1 that $E \cap F : K$ is separable.
3. Suppose that $\text{char}(K) = p > 0$ and that $L : K$ is a totally inseparable algebraic extension (thus, every element of $L \setminus K$ is inseparable). Show that whenever $\alpha \in L$, then there is a non-negative integer n and an element $\theta \in K$ having the property that $m_\alpha(K) = t^{p^n} - \theta$.
Solution: Suppose that $\alpha \in L$. Then $m_\alpha(K)$ is an irreducible polynomial over K, so by question 4(a) has the shape $g(t^{p^n})$ for some non-negative integer n and an irreducible separable polynomial g. Suppose that g has degree 2 or more, and that its distinct roots in $\overline{K}$ are $\beta_1, \ldots, \beta_d$. Then for some index i one has $\beta_i = \alpha^{p^n}$ and $m_{\beta_i}(K) = g(t)$, by the irreducibility of

g. But then $\beta_i \in L$ is separable, because g is separable, contradicting the totally inseparable property of the extension $L : K$. It follows that g must have degree 1, and hence $m_\alpha(K) = t^{p^n} - \theta$, where $\theta = \alpha^{p^n} \in K$.