GALOIS THEORY: SOLUTIONS TO HOMEWORK 5

1. Suppose that $L : F$ and $L : F'$ are finite extensions with $F \subseteq L$ and $F' \subseteq L$, and further that $\psi : F \to F'$ is an isomorphism. Explain why there are at most $[L : F]$ ways to extend ψ to a homomorphism from L into L. [This is Corollary 3.6 – consider F-homomorphisms acting on L.]

   Solution: We apply the argument of the proof of Theorem 3.5, writing $K_0 = F$ and $K_0' = F'$, and taking $\sigma_0 = \psi$ as the isomorphism mapping K_0 into K_0' in place of the identity map. The remainder of the proof of Theorem 3.5 now remains identical, and shows that there are at most $[L : F]$ ways of extending $\sigma_0 = \psi$ to a homomorphism from L into L, as required.

2. Let M be a field. Show that the following are equivalent:
   (i) the field M is algebraically closed;
   (ii) every non-constant polynomial $f \in M[t]$ factors in $M[t]$ as a product of linear factors;
   (iii) every irreducible polynomial in $M[t]$ has degree 1;
   (iv) the only algebraic extension of M containing M is M itself.

   Solution: Suppose that (i) holds. Consider $f \in M[t] \setminus M$, and note that f has a root $\alpha_1 \in M$. With $n = \deg f$, we define g_i inductively as follows. Define $g_1 \in M[t]$ by means of the relation $f = (t - \alpha_1)g_1$. Then, for $1 < i \leq n$, define $g_i \in M[t]$ by means of the relation $g_{i-1} = (t - \alpha_i)g_i$. Since $\deg g_i = n - i$, we see that g_{i-1} is non-constant for $1 < i \leq n$, and hence has a root $\alpha_i \in M$. We note in this context that $g_n \in M^\times$ is the leading coefficient of f. Thus $f = g_n(t - \alpha_1) \cdots (t - \alpha_n)$, and thus (i) implies (ii).
   Suppose next that (ii) holds, and suppose that $f \in M[t]$ is irreducible. Then f is non-zero and non-constant. Since f factors as a product of $\deg f$ linear factors, we must have $\deg f = 1$, and thus (ii) implies (iii).
   Next suppose that (iii) holds, and suppose that α lies in some algebraic extension field N extending M. Assume $M \subseteq N$. Then α is algebraic over M, and hence there is some irreducible polynomial $m_\alpha(M) \in M[t]$, which, in view of the hypothesis (iii), has degree 1. Since this polynomial is also monic, we infer that $t - \alpha = m_\alpha(M) \in M[t]$, whence $\alpha \in M$. But then $N = M$, and so (iii) implies (iv).
   Finally, suppose that (iv) holds. Let $f \in M[t] \setminus M$, and let N be a field extension of M with $M \subseteq N$ containing a root α of f. Then $M(\alpha) : M$ is an algebraic extension. The hypothesis (iv) thus implies that $M(\alpha) = M$, whence $\alpha \in M$. Then (iv) implies (i).
   In this way, we have confirmed the equivalence of (i), (ii), (iii) and (iv).

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