GALOIS THEORY: SOLUTIONS TO HOMEWORK 6

1. Suppose that L and M are fields with an associated homomorphism $\psi : L \to M$. Show that whenever L is algebraically closed, then $\psi(L)$ is also algebraically closed.

   Solution: Suppose that L is algebraically closed, and that $f' \in \psi(L)[t]$ is irreducible. Then we have $f' = \psi(f)$ for some $f \in L[t]$, and $\deg f' = \deg f$. For the sake of deriving a contradiction, suppose that $\deg f' > 1$. Then $\deg f > 1$. Since L is algebraically closed, it follows that irreducible polynomials in $L[t]$ have degree 1. We are forced to conclude, therefore, that f is reducible, and hence that $f = gh$ for some polynomials $g, h \in L[t]$ with $\deg g \geq 1$ and $\deg h \geq 1$. Consequently, we have $f' = g'h'$, where $g' = \psi(g)$ and $h' = \psi(h)$ satisfy the property that $\deg g' \geq 1$ and $\deg h' \geq 1$. However, this contradicts the assumption that f' is irreducible in $\psi(L)[t]$. We must therefore have $\deg f' = 1$. Thus we conclude that $\psi(L)$ is algebraically closed.

2. Let $L : K$ be a field extension with $K \subseteq L$. Let $\gamma \in L$ be transcendental over K, and consider the simple field extension $K(\gamma) : K$. Show that $K(\gamma)$ is not algebraically closed.

   Solution: Put $M = K(\gamma)$, and suppose that M is algebraically closed. We show that the polynomial $t^2 - \gamma$ is irreducible over $M[t]$, contradicting that M is algebraically closed, and thereby establishing the desired conclusion. Suppose then that $\alpha \in M$ satisfies the relation $\alpha^2 = \gamma$. Since $\alpha \in M = K(\gamma)$, it follows that there exists $n, m \in \mathbb{Z}_{\geq 0}$ and $a_i, b_i \in K$ $(0 \leq i \leq n)$, with $a_n \neq 0$ and $b_m \neq 0$, having the property that

$$\alpha = \frac{a_0 + a_1\gamma + \ldots + a_n\gamma^n}{b_0 + b_1\gamma + \ldots + b_m\gamma^m},$$

   whence

$$(a_0 + a_1\gamma + \ldots + a_n\gamma^n)^2 = \gamma(b_0 + b_1\gamma + \ldots + b_m\gamma^m)^2.$$

   Hence

$$a_n^2\gamma^{2n} + \ldots + a_0^2 = b_m^2\gamma^{2m+1} + \ldots + b_0^2\gamma.$$

   Either $2n > 2m + 1 \geq 1$, in which case γ is a root of the polynomial

$$a_n^2 t^{2n} + \ldots + a_0^2 \in K[t] \setminus K,$$

   or else $2m + 1 > 2n \geq 0$, in which case γ is a root of the polynomial

$$b_m^2 t^{2m+1} + \ldots - a_0^2 \in K[t] \setminus K.$$

   We therefore deduce that γ is algebraic over K, contradicting our hypotheses that γ is transcendental over K. Thus $K(\gamma)$ cannot be algebraically closed.