

## NUMBER THEORY: HOMEWORK 13

TO BE HANDED IN BY THURSDAY 24TH APRIL 2025 BY 6PM

**1.** Let  $k$  be a positive integer.

(a) Find the fundamental solution of the Pell equation  $x^2 - (k^2 + 2)y^2 = 1$ , and hence write down a relation that determines all solutions of this equation.

(b) Find two distinct solutions  $(x_1, y_1)$  and  $(x_2, y_2)$  of the generalised Pell equation  $x^2 - (k^2 + 2)y^2 = -2$  in which all  $x_i$  and  $y_i$  are positive.

**2.** Let  $d$  be a positive integer which is not a perfect square, and consider the negative Pell equation

$$x^2 - dy^2 = -1.$$

(a) Suppose that this equation has distinct non-trivial solutions  $(x_1, y_1)$  and  $(x_2, y_2)$  with respective associated real numbers  $x_1 + y_1\sqrt{d}$  and  $x_2 + y_2\sqrt{d}$ . Show that the real number

$$x_3 + y_3\sqrt{d} = \frac{x_2 + y_2\sqrt{d}}{x_1 + y_1\sqrt{d}}$$

is associated with a solution  $(x_3, y_3)$  of the Pell equation  $x^2 - dy^2 = 1$ .

(b) Suppose that the fundamental solution of the Pell equation  $x^2 - dy^2 = 1$  is  $(x_0, y_0)$ . Find a relation that defines *all* solutions of the negative Pell equation  $x^2 - dy^2 = -1$ .

**3.** Let  $k$  be a positive integer.

(a) Find the fundamental solution of the Pell equation  $x^2 - (k^2 + 1)y^2 = 1$ , and hence write down a relation that determines all solutions of this equation.

(b) Find a solution of the negative Pell equation  $x^2 - (k^2 + 1)y^2 = -1$ , and hence write down a relation that determines all solutions of this equation.

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