SOLUTIONS TO HOMEWORK 10

1. (a) Suppose that n and m are coprime with $n = \prod_{p^h \| n} p^h$ and $m = \prod_{\pi^h | m} \pi^h$, say, with p and π denoting prime numbers. Since $(n, m) = 1$, the primes p and π occurring in these products are distinct, and thus

$$s(nm) = \prod_{p|nm} p = \left(\prod_{p|n} p \right) \left(\prod_{\pi|m} \pi \right) = s(n)s(m).$$

Moreover, one has $s(1) = 1$, and so we conclude that $s(n)$ is a multiplicative function of n.

(b) By Möbius inversion, the arithmetic function $f(n)$ defined by putting

$$f(n) = \sum_{d|n} \mu(d)s(n/d)$$

satisfies the property that $s(n) = \sum_{d|n} f(d)$. But $\mu(n)$ and $s(n)$ are both multiplicative functions, and thus f is also a multiplicative function. We have $f(1) = 1$, and when p is prime and $h \geqslant 1$,

$$f(p^h) = \sum_{a=0}^{h} \mu(p^a)s(p^{h-a}) = s(p^h) - s(p^{h-1}) = \begin{cases} p - 1, & \text{when } h = 1, \\ p - p = 0, & \text{when } h \geqslant 2. \end{cases}$$

Thus, in all cases one has $f(p^h) = \mu^2(p^h)\varphi(p^h)$, and by multiplicativity we conclude that $f(n) = \mu^2(n)\varphi(n)$.

2. For each prime power p^h one has

$$\begin{aligned} \sum_{j=0}^{h} \phi(p^{h-j})\tau(p^j) &= \sum_{j=0}^{h-1} (p^{h-j} - p^{h-j-1})(j + 1) + \phi(p^0)\tau(p^h) \\ &= p^h + p^{h-1} + \cdots + p - h + h + 1 = \sum_{d|p^h} d = \sigma(p^h), \end{aligned}$$

and so

$$\sigma(p^h) = \sum_{d|p^h} \phi(p^h/d)\tau(d).$$

Thus it follows from multiplicativity of left and right hand sides that $\sigma(n) = \sum_{d|n} \phi(n/d)\tau(d)$ for $n \in \mathbb{N}$.

3. (a) We have $\sigma(n) = \sum_{d|n} d$, and hence

$$\begin{aligned}\sum_{1\leqslant n\leqslant x} \frac{\sigma(n)}{n^2} &= \sum_{1\leqslant n\leqslant x} \sum_{d|n} \frac{d}{n^2} = \sum_{1\leqslant d\leqslant x} \sum_{1\leqslant m\leqslant x/d} \frac{d}{(md)^2} \\ &= \sum_{1\leqslant d\leqslant x} \frac{1}{d} \sum_{1\leqslant m\leqslant x/d} \frac{1}{m^2} = \sum_{1\leqslant d\leqslant x} \frac{1}{d}\left(\frac{\pi^2}{6} + O(d/x)\right) \\ &= \frac{\pi^2}{6} \sum_{1\leqslant d\leqslant x} \frac{1}{d} + O\left(\frac{1}{x}\sum_{1\leqslant d\leqslant x} 1\right) \\ &= \frac{\pi^2}{6}\log x + O(1).\end{aligned}$$

(b) Also, we have $\phi(n) = n\sum_{d|n} \mu(d)/d$, and hence

$$\begin{aligned}\sum_{1\leqslant n\leqslant x} \frac{\phi(n)}{n^2} &= \sum_{1\leqslant n\leqslant x} \frac{1}{n} \sum_{d|n} \mu(d)/d = \sum_{1\leqslant d\leqslant x} \sum_{1\leqslant m\leqslant x/d} \frac{\mu(d)}{md^2} \\ &= \sum_{1\leqslant m\leqslant x} \frac{1}{m} \sum_{1\leqslant d\leqslant x/m} \frac{\mu(d)}{d^2} = \sum_{1\leqslant m\leqslant x} \frac{1}{m}\left(\frac{6}{\pi^2} + O(m/x)\right) \\ &= \frac{6}{\pi^2}\log x + O(1).\end{aligned}$$

4. (a) By multiplicativity, one has

$$\frac{\mu^2(d)}{d^2} = \prod_{p|d} p^{-2},$$

when d is squarefree, and $\mu^2(d)/d^2 = 0$ otherwise, and hence

$$\prod_p (1 + 1/p^2) = \sum_{\substack{d=1 \\ d \text{ squarefree}}}^{\infty} \prod_{p|d} p^{-2} = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2}.$$

(b) Then

$$\sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} = \frac{\prod_p (1 - 1/p^2)^{-1}}{\prod_p (1 - 1/p^4)^{-1}} = \frac{\zeta(2)}{\zeta(4)}.$$

Thus

$$\sum_{1\leqslant d\leqslant x} \frac{\mu^2(d)}{d^2} = \sum_{d=1}^{\infty} \frac{\mu^2(d)}{d^2} + O\Big(\sum_{d>x} \frac{1}{d^2}\Big) = \frac{\pi^2/6}{\pi^4/90} + O(1/x) = \frac{15}{\pi^2} + O(1/x).$$

5. Define

$$\beta(n) = \sum_{\substack{a=1 \\ (a,n)=1}}^{n} \log(a/n).$$

Then, applying the formula that we are asked to recall in the question, one obtains

$$\beta(n) = \sum_{d|n} \sum_{\substack{a=1 \\ d|a}}^{n} \mu(d) \log(a/n) = \sum_{d|n} \mu(d) \sum_{b=1}^{n/d} \log\left(\frac{b}{n/d}\right)$$

$$= \sum_{d|n} \mu(n/d) \sum_{b=1}^{d} \log(b/d) = \sum_{d|n} \mu(n/d) \log(d!/d^d).$$

Consequently, one finds that

$$\prod_{\substack{a=1 \\ (a,n)=1}}^{n} a = n^{\phi(n)} e^{\beta(n)} = n^{\phi(n)} \prod_{d|n} \left(\frac{d!}{d^d}\right)^{\mu(n/d)}.$$