SOLUTIONS TO HOMEWORK 13

1. (a) We begin by finding the continued fraction expansion of $\sqrt{k^2+2}$, noting that

$$[\sqrt{k^2+2}] = k, \quad 1/(\sqrt{k^2+2}-k) = \tfrac{1}{2}(\sqrt{k^2+2}+k),$$

$$[\tfrac{1}{2}(\sqrt{k^2+2}+k)] = k, \quad 1/(\tfrac{1}{2}(\sqrt{k^2+2}+k)-k) = 2/(\sqrt{k^2+2}-k) = \sqrt{k^2+2}+k,$$

$$[\sqrt{k^2+2}+k] = 2k, \quad 1/((\sqrt{k^2+2}+k)-2k) = 1/(\sqrt{k^2+2}-k) = \tfrac{1}{2}(\sqrt{k^2+2}+k),$$

and we obtain repetition. Thus $\sqrt{k^2+2} = [k; \overline{k, 2k}]$.

This is a continued fraction, periodic with period 2, and thus the fundamental solution corresponds to the convergent $p_1/q_1 = (k\cdot k+1)/k$ of the continued fraction expansion of $\sqrt{k^2+2}$. Indeed, one has $(k^2+1)^2 - (k^2+2)k^2 = 1$. Then the fundamental solution of this Pellian equation $x^2-(k^2+2)y^2 = 1$ is $(x,y) = (k^2+1, k)$. All solutions are determined by the relation

$$x + y\sqrt{k^2+2} = \pm(k^2+1+k\sqrt{k^2+2})^n \quad (n \in \mathbb{Z}).$$

(b) One solution immediately visible to the equation $x^2-(k^2+2)y^2 = -2$ is $(x_1, y_1) = (k, 1)$. We can find a second by examining the integers (x_2, y_2) defined via the relation

$$\begin{aligned} x_2 + y_2\sqrt{k^2+2} &= (k+\sqrt{k^2+2})(k^2+1+k\sqrt{k^2+2}) \\ &= k(k^2+1) + k(k^2+2) + (k^2+1+k^2)\sqrt{k^2+2}. \end{aligned}$$

Indeed, if we take $(x_2, y_2) = (2k^3+3k, 2k^2+1)$, then we find that

$$x_2^2 - (k^2+2)y_2^2 = (2k^3+3k)^2 - (k^2+2)(2k^2+1)^2 = -2.$$

2. (a) Observe that

$$\begin{aligned} x_3 + y_3\sqrt{d} = \frac{x_2+y_2\sqrt{d}}{x_1+y_1\sqrt{d}} &= \frac{(x_2+y_2\sqrt{d})(x_1-y_1\sqrt{d})}{x_1^2-dy_1^2} \\ &= -(x_1x_2 - dy_1y_2 + (x_1y_2 - x_2y_1)\sqrt{d}). \end{aligned}$$

Moreover, one has

$$\begin{aligned} x_3^2 - dy_3^2 &= (x_1x_2 - dy_1y_2)^2 - d(x_1y_2 - x_2y_1)^2 \\ &= (x_1^2 - dy_1^2)(x_2^2 - dy_2^2) = (-1)^2 = 1. \end{aligned}$$

(b) Suppose that (x_1, y_1) is the solution from (a) of the equation $x^2 - dy^2 = -1$. Then, whenever (x_2, y_2) is a second solution, it follows from part (a) that $x_2 + y_2\sqrt{d} = (x_1 + y_1\sqrt{d})(x_3 + y_3\sqrt{d})$ for some solution (x_3, y_3) of the Pell equation $x^2 - dy^2 = 1$. But by considering the general solution of the Pellian equation $x^2 - dy^2 = 1$, one must have $x_3 + y_3\sqrt{d} = \pm(x_0 + y_0\sqrt{d})^n$

for some $n \in \mathbb{Z}$. Hence, the set of all solutions of the negative Pell equation $x^2 - dy^2 = -1$ is given by the relation

$$x + y\sqrt{d} = \pm(x_1 + y_1\sqrt{d})(x_0 + y_0\sqrt{d})^n \quad (n \in \mathbb{Z}).$$

3. (a) We begin by finding the continued fraction expansion of $\sqrt{k^2+1}$, noting that

$$[\sqrt{k^2+1}] = k, \quad 1/(\sqrt{k^2+1} - k) = \sqrt{k^2+1} + k,$$

$$[\sqrt{k^2+1}+k] = 2k, \quad 1/((\sqrt{k^2+1}+k)-2k) = 1/(\sqrt{k^2+1}-k) = \sqrt{k^2+1}+k,$$

and we obtain repetition. Thus $\sqrt{k^2+1} = [k; \overline{2k}]$.

This is a continued fraction, periodic with period 1, and thus the fundamental solution corresponds to the convergent $p_1/q_1 = (k \cdot 2k+1)/2k$ of the continued fraction expansion of $\sqrt{k^2+1}$. Indeed, one has $(2k^2+1)^2 - (k^2+1)(2k)^2 = 1$. Then the fundamental solution of this Pellian equation $x^2 - (k^2+1)y^2 = 1$ is $(x, y) = (2k^2+1, 2k)$. All solutions are determined by the relation

$$x + y\sqrt{k^2+1} = \pm(2k^2 + 1 + 2k\sqrt{k^2+1})^n \quad (n \in \mathbb{Z}).$$

(b) A self-evident solution of the negative Pell equation $x^2 - (k^2+1)y^2 = -1$ is $(x, y) = (k, 1)$. Thus, in view of the solution of problem 2(b) together with problem 3(a), the general solution of this negative Pell equation is furnished by the relation

$$x + y\sqrt{k^2+1} = \pm(k + \sqrt{k^2+1})(2k^2 + 1 + 2k\sqrt{k^2+1})^n \quad (n \in \mathbb{Z}).$$

We note that $(k + \sqrt{k^2+1})^2 = 2k^2 + 1 + 2k\sqrt{k^2+1}$, and thus the previous relation may be abbreviated to the relation

$$x + y\sqrt{k^2+1} = \pm(k + \sqrt{k^2+1})^{2n+1} \quad (n \in \mathbb{Z}).$$

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