Twins of k-free numbers and their exponential sum

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1. Introduction. For any integer $k \geq 2$ let $\mu_k(n)$ denote the characteristic function on the set of k-free numbers, that is, $\mu_k(n) = 0$ if there is a prime p with $p^k|n$, and $\mu_k(n) = 1$ otherwise. A twin of k-free numbers is a natural number n such that $\mu_k(n) = \mu_k(n+1) = 1$. It has long been known that the set of these twins has positive density

$$\varrho = \varrho_k = \prod_p \Big(1 - \frac{2}{p^k}\Big), \tag{1.1}$$

and although the first explicit reference to an asymptotic formula for the counting function

$$A_k(x) = \sum_{n \leq x} \mu_k(n)\mu_k(n+1)$$

seems to be a paper by Carlitz [2], the estimate

$$A_k(x) = \varrho x + O(x^{\frac{2}{k+1}+\varepsilon}) \tag{1.2}$$

is at least implicit in the work of Evelyn and Linfoot [4] and Estermann [3]. The latter formula (1.2) was then proved in refined form, with x^ε replaced by $(\log x)^{4/3}$, by Mirsky [7]. More recently, Heath-Brown [5] considered the case $k = 2$ and obtained (1.2) with $O(x^{\frac{7}{11}+\varepsilon})$ in place of $O(x^{\frac{2}{3}+\varepsilon})$.

In this paper we study the exponential sum

$$S(\alpha) = S_k(\alpha) = \sum_{n \leq x} \mu_k(n)\mu_k(n+1)e(\alpha n) \tag{1.3}$$

associated with k-free twins. In recent years there has been an increased interest in the L_1-norm of exponential sums over reasonably dense sets of which the k-free twins form an example. Our first theorem adds to the small stock of such sums for which a non-trivial estimate can be obtained.

Theorem 1. *Let $k \geq 2$. Then*

$$\int_0^1 |S_k(\alpha)| d\alpha \ll x^{\frac{1}{k+1}+\varepsilon}.$$

The trivial upper bound for this integral is $O(\sqrt{x})$, and is obtained through Cauchy-Schwarz's inequality and Parseval's identity

$$\int_0^1 |S_k(\alpha)|^2 d\alpha = A_k(x). \tag{1.4}$$

According to general principles, the L_1-norm is bounded below by a function not much smaller than $\sqrt{x}$ if the underlying sequence is not well-distributed among a fair share of the arithmetic progressions. Inversely, if the sequence is well- and reasonably equi-distributed in most arithmetic progressions, then the L_2-norm (1.4) tends to be concentrated on the major arcs in a standard Hardy-Littlewood dissection of the unit interval. Not unexpectedly, the k-free twins fall into the latter category, as the next theorem shows.

Let $1 \le Q \le \frac{1}{2}\sqrt{x}$, and let $\mathfrak{M} = \mathfrak{M}(Q)$ denote the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [Q^{-1}, 1+Q^{-1}] \ : \ |q\alpha - a| < Q/x\}$$

with $1 \le a \le q \le Q$ and $(a,q) = 1$. Moreover, let $\mathfrak{m} = \mathfrak{m}(Q) = [Q^{-1}, 1 + Q^{-1}]\backslash\mathfrak{M}(Q)$. We have

Theorem 2. *Let $k \ge 2$. Then*

$$\int_{\mathfrak{m}(Q)} |S_k(\alpha)|^2 d\alpha \ll x^{1+\varepsilon} Q^{\frac{1}{k}-1} + Q^{3-\frac{2}{k}} x^{\frac{2}{k}-1+\varepsilon} + x^{\frac{4}{k+1}-1+\varepsilon} Q^2.$$

These estimates should be compared with the results of a recent investigation by Brüdern, Granville, Perelli, Vaughan and Wooley [1], hereafter cited as V, where the exponential sum over k-free numbers was studied. In particular, it was shown that one has

$$\int_0^1 \Big| \sum_{n \le x} \mu_k(n) e(\alpha n) \Big| d\alpha \ll x^{\frac{1}{k+1}+\varepsilon}, \tag{1.5}$$

$$\int_{\mathfrak{m}(Q)} \Big| \sum_{n \le x} \mu_k(n) e(\alpha n) \Big| d\alpha \ll x^{1+\varepsilon} Q^{\frac{1}{k}-1} + x^{\frac{2}{k}-1+\varepsilon} Q^{3-\frac{2}{k}}. \tag{1.6}$$

These estimates seem to be the first instances where L_1-norms and L_2-norms over minor arcs allowed for a breaking through the familiar "square root cancellation" barrier, leaving aside trivial examples such as arithmetic progressions. The results of this paper show that such is possible even if the underlying sequence is not multiplicative. We refer the reader to Perelli [8] for a more exhaustive survey of this matter.

Note that the estimates (1.5) and in Theorem 1 are of the same strength. The proof of (1.5) in V is elementary and depends mainly on the convolution formula

$$\mu_k(n) = \sum_{d^k | n} \mu(d). \tag{1.7}$$

In the new context of twins we make use of (1.7) for n and $n+1$. By Schwarz's inequality, applied to a suitable portion of the resulting exponential sum, it is possible to link the L_1-norm of $S(\alpha)$ to an upper bound for the number of solutions of the diophantine equation $sv^k - ru^k = 1$, with all four variables in certain ranges. An elaboration of the ideas of V then leads to Theorem 1. We present the details in §2.

Theorem 2 compares easily with the very similar bound (1.6). The strategy is the same as in V, though in the present context we must examine the distribution of k-free twins in arithmetic progression. By standard methods, this information can be transported to an asymptotic formula for the major arc contribution to (1.4). This takes the shape

$$\int_{\mathfrak{M}} |S(\alpha)|^2 d\alpha \sim \varrho^2 \mathfrak{S} x \tag{1.8}$$

where $\mathfrak{S}$ is the singular series associated naturally with the trivial equation $n = m$ in k-free twins (see (4.4) for a precise definition). A comparison of Euler products shows that $\mathfrak{S} = \varrho^{-1}$, and from (1.8), (1.4) and (1.2) one finds

$$\int_{\mathfrak{m}} |S(\alpha)|^2 d\alpha = o(x)$$

as $x \to \infty$. Explicit control of error terms in this argument yields Theorem 2.

As in V one can deduce from Theorem 2 results for binary additive problems with twins of k-free numbers. We content ourselves with just one example. For $k \geq l \geq 2$, let

$$r_{k,l}(n) = \sum_{a+b=n} \mu_k(a)\mu_k(a+1)\mu_l(b)\mu_l(b+1)$$

denote the number of representations of n as the sum of a k-free twin and an l-free twin. Let $\mathfrak{S}_{k,l}(n)$ denote the natural singular series associated with this binary problem (see (6.6) for a definition). We have

Theorem 3. *Let $k \geq l \geq 2$. Then*

$$r_{k,l}(n) = \mathfrak{S}_{k,l}(n)\varrho_k\varrho_l n + O(n^{9/10+\varepsilon}).$$

We are certainly not asserting that this asymptotic formula could not be obtained by an elementary argument, or that the error term is the sharpest obtainable one. The point is the relative ease by which the result is obtained, and that the circle method succeeds at all with a binary additive problem, contrary to a widely held belief. As we shall see in section 6, the circle method neatly disentangles the different multiplicative constraints on the two summands.

One might ask whether the results of this paper persist in more general situations such as r-tuples of k-free numbers, that is, integers n such that $n, n+b_1, \ldots, n+b_{r-1}$ are all k-free. This is indeed the case, and at least this particular example can be treated by the ideas in this paper, at the cost of extra complication in detail. The arguments in §2 may be extended to establish the bound

$$\int_0^1 \Big| \sum_{n \leq x} \mu_k(n)\mu_k(n+b_1)\ldots\mu_k(n+b_{r-1})e(\alpha n)\Big| d\alpha \ll x^{\frac{1}{k+1}+\varepsilon}.$$

Similarly, the conclusions of Theorem 2 can be validated for exponential sums over r-tuples, by working along the lines of Tsang [9]. There is, however, a grander design underneath the surface of the present article which relates the study of exponential sums cognate to their prototype (1.3) with a sieve theory which we hope to present in a forthcoming publication.

Our notation is standard or otherwise explained at the appropriate stage of the argument. Statements involving an ε are true for all $\varepsilon > 0$, with implicit constants in Vinogradov or Landau symbols depending on ε.

2. The L_1-norm. We prepare for the proof of Theorem 1 with a simple lemma which will also be of use in the next section where we deal with the distribution of k-free numbers in arithmetic progressions.

LEMMA 2.1. *Let $1 \le y \le x^{2/k}$, and let $\Theta(x,y)$ denote the number of quadruples r,s,u,v satisfying the conditions*

$$sv^k - ru^k = 1, \quad ru^k \le x \tag{2.1}$$

and $uv \ge y$. Then

$$\Theta(x,y) \ll x^{2+\varepsilon} y^{-k}, \tag{2.2}$$

and also

$$\Theta(x,y) \ll x^{1+\varepsilon} y^{1-k} + x^{\frac{2}{k+1}+\varepsilon}. \tag{2.3}$$

Proof. From (2.1) we have $ru^k sv^k \le x(x+1)$, whence $rs \le x(x+1)y^{-k}$ for any quadruple counted by $\Theta(x,y)$. The total number of choices for r,s therefore is bounded by $O(x^{2+\varepsilon}y^{-k})$, by a divisor argument. For any such choice of r,s, the number of solutions in u,v of the equation $sv^k - ru^k = 1$ is $O(x^\varepsilon)$ (see, for example, Estermann [3]), and (2.2) follows.

To derive (2.3), we note that for $y \ge x^{\frac{2}{k+1}}$, one has $x^2y^{-k} \le x^{\frac{2}{k+1}}$, whence (2.3) follows from (2.2). Therefore, we may suppose that $y < x^{\frac{2}{k+1}}$. Then, counting those quadruples where $uv > x^{\frac{2}{k+1}}$ again by (2.2), we find that

$$\Theta(x,y) \ll x^{\frac{2}{k+1}+\varepsilon} + \Theta^*,$$

where Θ^* is the number of quadruples r,s,u,v satisfying (2.1) and

$$y \le uv \le x^{\frac{2}{k+1}}.$$

From (2.1) one has $(u,v) = 1$. For any fixed choice of u,v, it follows that $ru^k \equiv -1 \pmod{v^k}$ which fixes the value of r modulo v^k. By (2.1), the total number of possibilities for r is $O(1 + x(uv)^{-k})$. But for any given r,u,v, the value of s is fixed by the equation in (2.1). Hence,

$$\Theta^* \ll \sum_{y \le uv \le x^{2/(k+1)}} (1 + x(uv)^{-k}) \ll x^{\frac{2}{k+1}+\varepsilon} + x^{1+\varepsilon}y^{1-k},$$

which implies (2.3).

The proof of Theorem 1 is now swiftly overwhelmed. To simplify notational obstacles, let $I(r,s,u,v)$ denote the condition that r,s,u,v satisfy (2.1). Then, by (1.3) and the convolution formula (1.7), imported for $\mu_k(n)$ and $\mu_k(n+1)$, we infer that

$$S(\alpha) = \sum_{I(r,s,u,v)} \mu(u)\mu(v)e(\alpha r u^k).$$

Let $1 \le y \le x^{2/k}$, and write

$$T_1(\alpha) = \sum_{\substack{I(r,s,u,v)\\ uv\le y}} \mu(u)\mu(v)e(\alpha r u^k), \quad T_2(\alpha) = \sum_{\substack{I(r,s,u,v)\\ uv> y}} \mu(u)\mu(v)e(\alpha r u^k).$$

Then, by Schwarz's inequality,

$$\int_0^1 |S(\alpha)|\, d\alpha \le \int_0^1 |T_1(\alpha)|\, d\alpha + \Big(\int_0^1 |T_2(\alpha)|^2\, d\alpha\Big)^{\frac12}. \tag{2.4}$$

To estimate the second summand on the right hand side, we observe that the number of quadruples r,s,u,v satisfying (2.1) with a prescribed value of ru^k is $O(x^\varepsilon)$, by an immediate divisor argument. Hence, by Parseval's identity and Lemma 2.1,

$$\int_0^1 |T_2(\alpha)|^2\, d\alpha \ll x^\varepsilon \Theta(x,y) \ll x^{1+\varepsilon}y^{1-k} + x^{\frac{2}{k+1}+\varepsilon}.$$

The treatment of the first term on the right hand side of (2.4) is different. We pick up the condition $sv^k = ru^k + 1$ implicit in $I(r,s,u,v)$ by orthogonality and rewrite $T_1(\alpha)$ as

$$T_1(\alpha) = \sum_{uv\le y} \mu(u)\mu(v) \int_0^1 V((\alpha+\beta)u^k, xu^{-k})V(-\beta v^k, (x+1)v^{-k})e(\beta)\, d\beta,$$

where

$$V(\gamma, z) = \sum_{m\le z} e(\gamma m).$$

It follows that

$$\int_0^1 |T_1(\alpha)|\, d\alpha \le \sum_{uv\le y} \int_0^1\int_0^1 |V((\alpha+\beta)u^k, xu^{-k})V(-\beta v^k, (x+1)v^{-k})|\, d\alpha\, d\beta.$$

The function $V(\gamma, z)$ has period 1 in γ. By a change of variable, we infer that

$$\begin{aligned}\int_0^1 |T_1(\alpha)|\, d\alpha &\le \sum_{uv\le y} \int_0^1\int_0^1 |V((\alpha+\beta), xu^{-k})V(-\beta, (x+1)v^{-k})|\, d\alpha\, d\beta \\ &\ll \sum_{uv\le y} \int_0^1\int_0^1 \min(x, \|\alpha+\beta\|^{-1})\min(x, \|\beta\|^{-1})\, d\alpha\, d\beta \\ &\ll y(\log y)(\log x)^2.\end{aligned}$$

Choosing $y = x^{1/(k+1)}$, Theorem 1 now follows from (2.4).

3. Twins of k-free numbers in arithmetic progressions. The relevance of the distribution in arithmetic progressions for the success of our method has already been stressed. Neither an asymptotic formula for the counting function

$$A_k(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \mu_k(n)\mu_k(n+1) \tag{3.1}$$

nor an estimate for the variance of the ensuing error terms seem to be available in the literature. We therefore proceed by supplying such formulae. Let

$$g(q, a) = \sum_{\substack{u,v=1 \\ (u^k, q) | a \\ (v^k, q) | a+1}}^{\infty} \frac{\mu(uv)}{u^k v^k}(q, u^k v^k). \tag{3.2}$$

We then have the following elementary estimates.

Lemma 3.1. *Uniformly in a and q, one has*

$$A_k(x; q, a) = q^{-1} g(q, a)x + O(x^{\frac{2}{k+1}+\varepsilon}).$$

Once a main term for $A_k(x; q, a)$ has been determined, it is natural to consider the variance

$$\Upsilon_k(x, Q) = \sum_{q \le Q} \sum_{a=1}^{q} \left| A_k(x; q, a) - q^{-1} g(q, a)x \right|^2.$$

Lemma 3.2. *When $1 \le Q \le x$, one has*

$$\Upsilon_k(x, Q) \ll x^{\frac{2}{k}+\varepsilon} Q^{2-\frac{2}{k}} + x^{\frac{4}{k+1}+\varepsilon}.$$

Both lemmata follow from a common principle. We continue to use the notational conventions introduced in §2. Then, writing $n = ru^k$, $n+1 = sv^k$ in (3.1), we infer from (1.7) that

$$A_k(x; q, a) = \sum_{\substack{I(r,s,u,v) \\ ru^k \equiv a \pmod q}} \mu(u)\mu(v) = B_1(q, a) + B_2(q, a), \tag{3.3}$$

say, where $B_1(q, a)$ is the portion of the central sum with $uv \le y$, and $B_2(q, a)$ is the complementary part with $uv > y$. Here $1 \le y \le x^{2/k}$ is a parameter at our disposal.

We evaluate $B_1(q, a)$ by counting, for any given pair u, v with $uv \le y$, the number of r, s such that $ru^k \equiv a \pmod q$, and $I(r, s, u, v)$ holds. From $sv^k - ru^k = 1$ one has $(u, v) = 1$. Moreover, the congruences $ru^k \equiv a \pmod q$

and $sv^k \equiv a+1 \pmod q$ imply that $(u^k,q)|a$ and $(v^k,q)|a+1$. Thus, the simultaneous conditions $I(r,s,u,v)$ and $ru^k \equiv a \pmod q$ necessitate that

$$(u,v)=1, \quad (u^k,q)|a, \quad (v^k,q)|a+1, \tag{3.4}$$

as we henceforth assume. Subject to these extra conditions, we note that for a given r there will be an integer s such that $sv^k - ru^k = 1$ if and only if $ru^k \equiv -1 \pmod{v^k}$.

Next, since $(u,v)=1$ and $a \equiv -1 \pmod{(q,v^k)}$, the simultaneous congruences

$$r\frac{u^k}{(u^k,q)} \equiv \frac{a}{(u^k,q)} \pmod{\frac{q}{(u^k,q)}} \quad \text{and} \quad ru^k \equiv -1 \pmod{v^k} \tag{3.5}$$

are compatible, and combine to a single congruence to modulus

$$\frac{qv^k}{(u^k,q)(v^k,q)}.$$

It follows that the congruences (3.5) have

$$\frac{x(q,u^k)(q,v^k)}{qu^kv^k} + O(1)$$

solutions r with $1 \le r \le xu^{-k}$, provided that (3.4) holds. It follows that

$$B_1(q,a) = \sum_{\substack{uv \le y \\ (3.4) \text{ holds}}} \left(\frac{x(q,u^kv^k)}{qu^kv^k}\mu(u)\mu(v) + O(1) \right).$$

Finally, we note that $\mu(uv)=0$ if $(u,v)>1$, so that we may replace $\mu(u)\mu(v)$ by $\mu(uv)$ and then drop $(u,v)=1$ from the summation condition. In order to complete the sum over $uv<y$ to an infinite series, we proceed as follows. We define for $1 \le i < k$ the integers

$$\pi_i = \prod_{p^i \| (q,a)} p, \quad \varpi_i = \prod_{p^i \| (q,a+1)} p,$$

and define also

$$\pi_k = \prod_{p^k | (q,a)} p, \quad \varpi_k = \prod_{p^k | (q,a+1)} p.$$

By invoking the simple bound

$$\begin{aligned} &\sum_{\substack{U<u\le 2U \\ (u^k,q)|a}} \sum_{\substack{V<v\le 2V \\ (v^k,q)|a+1}} \mu^2(uv)\frac{(q,u^kv^k)}{u^kv^k} \\ &\qquad \le (UV)^{-k} \sum_{\substack{e_1|\pi_1 \\ f_1|\varpi_1}} \cdots \sum_{\substack{e_k|\pi_k \\ f_k|\varpi_k}} \frac{UV}{e_1f_1\ldots e_kf_k} \prod_{i=1}^k (e_if_i)^i \\ &\qquad \ll q^\varepsilon (UV)^{1-k} \prod_{i=1}^k (\pi_i\varpi_i)^{i-1}, \end{aligned}$$

we find that

$$\sum_{\substack{uv>y \\ (u^k,q)|a \\ (v^k,q)|a+1}} \mu^2(uv)\frac{(q,u^kv^k)}{u^kv^k} \ll q^\varepsilon y^{1-k}\prod_{i=1}^{k}(\pi_i\varpi_i)^{i-1}$$

(the reader may care to compare this argument with that on pp. 744–745 of V). We then find that

$$B_1(q,a) - \frac{x}{q}\sum_{\substack{u,v=1 \\ (u^k,q)|a \\ (v^k,q)|a+1}}^{\infty} \frac{\mu(uv)}{u^kv^k}(q,u^kv^k) \ll y^{1+\varepsilon} + xy^{1-k+\varepsilon}q^{\varepsilon-1}\prod_{i=1}^{k}(\pi_i\varpi_i)^{i-1}.$$

This confirms the asymptotic formula

$$B_1(q,a) = xq^{-1}g(q,a) + O\left(y^{1+\varepsilon} + q^{\varepsilon-1}xy^{1-k+\varepsilon}\prod_{i=1}^{k}(\pi_i\varpi_i)^{i-1}\right). \tag{3.6}$$

To complete the proof of Lemma 3.1, we merely note that $B_2(q,a) \le \Theta(x,y)$, in the notation of Lemma 2.1. By (2.2), (3.6) and (3.3), it follows that

$$A_k(x;q,a) = xq^{-1}g(q,a) + O\left(x^{2+\varepsilon}y^{-k} + y^{1+\varepsilon}\right),$$

from which Lemma 3.1 is obtained by choosing $y = x^{\frac{2}{k+1}}$.

To derive Lemma 3.2, we observe that (3.6) and (3.3) yield

$$\begin{aligned}&|A_k(x;q,a) - q^{-1}g(q,a)x|^2 \\ &\qquad \ll y^{2+\varepsilon} + q^{\varepsilon-2}x^2y^{2-2k+\varepsilon}\prod_{i=1}^{k}(\pi_i\varpi_i)^{2i-2} + |B_2(q,a)|^2.\end{aligned}$$

Now

$$\sum_{a=1}^{q}|B_2(q,a)|^2 \le U(q),$$

where $U(q)$ is the number of r_j, s_j, u_j, v_j ($j = 1,2$) satisfying $I(r_j,s_j,u_j,v_j)$ for $j = 1,2$ and

$$r_1u_1^k \equiv r_2u_2^k \pmod q, \quad u_1v_1 > y, \quad u_2v_2 > y.$$

We sum over q and find that

$$\begin{aligned}\sum_{q\le Q}\sum_{a=1}^{q}|B_2(q,a)|^2 &\le \sum_{\substack{I^*(r_j,s_j,u_j,v_j) \\ j=1,2}}\ \sum_{\substack{q\le Q \\ q|r_1u_1^k-r_2u_2^k}} 1 \\ &\le Q\sum_{\substack{I^*(r_j,s_j,u_j,v_j) \\ r_1u_1^k=r_2u_2^k}} 1 + x^\varepsilon \sum_{\substack{I^*(r_j,s_j,u_j,v_j) \\ r_1u_1^k\ne r_2u_2^k}} 1\end{aligned}$$

where I^* indicates that I is supplemented by $uv > y$. For the first remaining sum we note that $r_1u_1^k = r_2u_2^k$ implies that $s_1v_1^k = s_2v_2^k$. Hence, if $I^*(r_1, s_1, u_1, v_1)$ holds, there are at most $O(x^\varepsilon)$ quadruples r_2, s_2, u_2, v_2 satisfying $I^*(r_2, s_2, u_2, v_2)$ and $r_1u_1^k = r_2u_2^k$. Hence, in the notation of the statement of Lemma 2.1,

$$\sum_{q\le Q}\sum_{a=1}^{q}|B_2(q,a)|^2 \ll Qx^\varepsilon\Theta(x,y) + x^\varepsilon\Theta(x,y)^2,$$

and therefore, by (2.3), and an argument similar to that straddling pp. 746 –747 of V,

$$\Upsilon_k(x,Q) \ll Q^2y^{2+\varepsilon} + x^{2+\varepsilon}y^{2-2k} + Q(x^{1+\varepsilon}y^{1-k} + x^{\frac{2}{k+1}+\varepsilon}) + x^{\frac{4}{k+1}+\varepsilon}$$

from which Lemma 3.2 follows by choosing $y = (x/Q)^{\frac{1}{k}}$.

For $Q > x$, a simple bound suffices for our needs. Since $g(q,a) \ll q^\varepsilon$ and $A_k(x;q,a) \le xq^{-1} + 1$ by obvious estimates, we have in this case

$$\Upsilon_k(x,Q) \ll \sum_{q\le Q}\sum_{a=1}^{q}\left(xq^{\varepsilon-1}\prod_{i=1}^{k}(\pi_i\varpi_i)^{i-1} + 1\right)^2 \ll x^2Q^\varepsilon + Q^2.$$

Perhaps it is worth pointing out that our approach to $\Upsilon_k(x,Q)$ is rather crude and susceptible to various improvements. When Q is small, the methods of Heath-Brown [5] and Tsang [9] will provide a better estimate, at least when $k = 2$. Indeed, when $k = 2$, Heath-Brown [5] has shown that $\Theta(x,y) \ll x^{7/6+\varepsilon}y^{-5/6}$ when $y > x^{1/2}$. Using this in the above argument, the error term in Lemma 3.1 may be reduced to $O(x^{7/11+\varepsilon})$, and also Lemma 3.2 may be improved in certain ranges of Q. Furthermore, the work of Vaughan [11] is likely to yield superior bounds when $\sqrt{x} < Q < x$. In the ranges for Q which are of interest in arithmetic applications such as Theorem 3, such improvements seem to have little impact.

A noticeable feature of our variance estimate is that the function $g(q,a)$ does not only depend on q and (a,q), unlike most sequences investigated hitherto. We draw the reader's attention to part X of Hooley's acclaimed series on this subject matter [6] where situations of this kind are analysed in an abstract set-up.

We close this section with a brief analysis of $g(q,a)$. By (3.2),

$$g(q,a) = \sum_{n=1}^{\infty}\mu(n)\frac{(q,n^k)}{n^k}\psi_k(n;q,a)$$

where $\psi_k(n;q,a)$ denotes the number of pairs u,v of natural numbers with $uv = n$ which satisfy (3.4). It is immediate that for any fixed a,q the function

$\psi_k(n;q,a)$ is multiplicative in n. Hence $g(q,a)$ can be written as an Euler product which takes the provisional form

$$g(q,a)=\prod_p\Big(1-\frac{(p^k,q)}{p^k}\psi_k(p;q,a)\Big).$$

By (3.4), we have $\psi_k(p;q,a)=2$ for all $p\nmid q$. It is therefore convenient to introduce the functions

$$f(q)=\prod_{p|q}\Big(1-\frac{2}{p^k}\Big)^{-1},\quad h(q,a)=\prod_{p|q}\Big(1-\frac{(p^k,q)}{p^k}\psi_k(p;q,a)\Big),\tag{3.7}$$

so that from (1.1) we can now infer the basic identity

$$g(q,a)=\varrho f(q)\,h(q,a).\tag{3.8}$$

For any $p|q$ let $p^\nu\|q$. Then $\psi_k(p;q,a)=\psi_k(p;p^\nu,a)$. The equation $uv=p$ admits the solutions $u=p,v=1$ and $u=1,v=p$. However, for $\nu\geq 1$, we cannot have $(p^k,p^\nu)|a$ and $(p^k,p^\nu)|a+1$ simultaneously. By (3.4), it follows that $\psi_k(p;p^\nu,a)=1$ if $(p^k,p^\nu)|a(a+1)$, and $\psi_k(p;p^k,a)=0$ otherwise. Consequently, we have

$$h(q,a)=\prod_{\substack{p^\nu\|q\\(p^\nu,p^k)|a(a+1)}}\Big(1-\frac{(p^k,p^\nu)}{p^k}\Big)=\prod_{\substack{p|q\\(p^k,q)|a(a+1)}}\Big(1-\frac{(p^k,q)}{p^k}\Big).\tag{3.9}$$

From this handier formula one readily confirms the quasi-multiplicative property that for any coprime natural numbers q_1,q_2 and any integers a_1,a_2 one has

$$h(q_1q_2,a_1q_2+a_2q_1)=h(q_1,a_1q_2)h(q_2,a_2q_1).\tag{3.10}$$

4. Gaussian sums and singular series. Recalling (3.2) and (3.7), we now form the sums of Gaussian type

$$G(q,a)=\sum_{b=1}^{q}g(q,b)e\Big(\frac{ab}{q}\Big),\quad H(q,a)=\sum_{b=1}^{q}h(q,b)e\Big(\frac{ab}{q}\Big)\tag{4.1}$$

which by (3.8) are related by

$$G(q,a)=\varrho f(q)H(q,a).\tag{4.2}$$

Then we introduce the sum

$$H(q)=\sum_{\substack{a=1\\(a,q)=1}}^{q}|H(q,a)|^2\tag{4.3}$$

which is used in turn to define the singular series

$$\mathfrak{S} = \sum_{q=1}^{\infty} q^{-2} f(q)^2 H(q). \tag{4.4}$$

LEMMA 4.1. *The function $H(q)$ is multiplicative. For all primes p one has*

$$\begin{aligned} H(p) &= 2p^{3-2k}\Big(1-\frac{2}{p}\Big), \\ H(p^\nu) &= 2p^{3\nu-2k}\Big(1-\frac{1}{p}\Big) \quad (2 \le \nu \le k), \\ H(p^\nu) &= 0 \quad (\nu > k). \end{aligned}$$

Proof. The multiplicative property follows from the Chinese Remainder Theorem, (3.10), (4.1) and (4.3) by a standard argument (see Vaughan [10], Lemma 2.11 for a model), and we may omit the details.

For any prime p and any $\nu \ge 1$, the orthogonality of characters and (4.1), (4.3) yield

$$\begin{aligned} H(p^\nu) &= \sum_{a=1}^{p^\nu} \Big| \sum_{b=1}^{p^\nu} h(p^\nu, b) e\Big(\frac{ab}{p^\nu}\Big)\Big|^2 - \sum_{\substack{a=1 \\ p|a}}^{p^\nu} \Big| \sum_{b=1}^{p^\nu} h(p^\nu, b) e\Big(\frac{ab}{p^\nu}\Big)\Big|^2 \\ &= p^\nu K_1(p^\nu) - p^{\nu-1} K_2(p^\nu) \end{aligned} \tag{4.5}$$

where

$$K_1(p^\nu) = \sum_{a=1}^{p^\nu} h(p^\nu, a)^2, \quad K_2(p^\nu) = \sum_{\substack{a,b=1 \\ a \equiv b \pmod{p^{\nu-1}}}}^{p^\nu} h(p^\nu, a) h(p^\nu, b). \tag{4.6}$$

We dispose of the case $\nu > k$ first. By (3.9), one has $h(p^\nu, a) = h(p^\nu, b)$ whenever $a \equiv b \pmod{p^k}$. Hence, for $\nu > k$,

$$K_2(p^\nu) = \sum_{a=1}^{p^\nu} h(p^\nu, a)^2 \sum_{\substack{b=1 \\ a \equiv b \bmod p^{\nu-1}}}^{p^\nu} 1 = pK_1(p^\nu),$$

and (4.5) yields $H(p^\nu) = 0$, as required.

We may now suppose that $1 \le \nu \le k$. By (3.9),

$$h(p^\nu, a) = \begin{cases} 1 - p^{\nu-k}, & \text{if } p^\nu | a(a+1), \\ 1, & \text{otherwise.} \end{cases} \tag{4.7}$$

From (4.6), we now find that

$$K_1(p^\nu) = 2(1-p^{\nu-k})^2 + \sum_{a=1}^{p^\nu-2} 1 = 2(1-p^{\nu-k})^2 + p^\nu - 2. \tag{4.8}$$

Similarly, when $\nu = 1$, we deduce from (4.6) and (4.7) that

$$K_2(p) = \Big(\sum_{a=1}^{p} h(p,a)\Big)^2 = \Big(2(1-p^{1-k}) + p - 2\Big)^2 = p^2(1-2p^{-k})^2.$$

When combined with (4.8) for $\nu = 1$, the identity $H(p) = 2p^{3-2k}(1-\frac{2}{p})$ is readily confirmed from (4.5).

It remains to consider the case where $2 \le \nu \le k$. By (4.6), terms with $a = b$ will contribute to $K_2(p^\nu)$ exactly $K_1(p^\nu)$. Hence, on writing

$$K_3(p^\nu) = \sum_{\substack{a,b=1\\ a\equiv b \pmod{p^{\nu-1}}\\ a\neq b}}^{p^\nu} h(p^\nu,a)h(p^\nu,b),$$

we infer from (4.5) that

$$H(p^\nu) = p^\nu\Big(1-\frac{1}{p}\Big)K_1(p^\nu) - p^{\nu-1}K_3(p^\nu). \tag{4.9}$$

A formula for $K_1(p^\nu)$ being already available, we proceed to evaluate $K_3(p^\nu)$. By (4.7), we have $h(p^\nu,a) = 1$ for $1 \le a \le p^\nu - 2$. We therefore split the sum $K_3(p^\nu)$ into the subsum, $K_4(p^\nu)$, where $1 \le a \le p^\nu - 2$ and $1 \le b \le p^\nu - 2$, and its complement, $K_5(p^\nu)$, where one at least of a and b is either $p^\nu - 1$ or p^ν. Now

$$\begin{aligned} K_4(p^\nu) &= \#\{(a,b) : 1 \le a,b \le p^\nu - 2,\, a\neq b,\, a\equiv b \pmod{p^{\nu-1}}\} \\ &= (p^\nu - 2p)(p-1) + (2p-2)(p-2) \\ &= (p^\nu - 4)(p-1). \end{aligned}$$

In order to evaluate $K_5(p^\nu)$, note that by the symmetry between a and b, one has

$$K_5(p^\nu) = 2\sum_{a=p^\nu-1}^{p^\nu} h(p^\nu,a) \sum_{\substack{b=1\\ b\equiv a \pmod{p^{\nu-1}}\\ b\neq a}}^{p^\nu} 1 = 4(1-p^{\nu-k})(p-1).$$

Since $K_3(p^\nu) = K_4(p^\nu) + K_5(p^\nu)$, we deduce from (4.8) and (4.9) and a straightforward computation that $H(p^\nu) = 2p^{3\nu-2k}(1-\frac{1}{p})$, as claimed. The proof of Lemma 3.1 is complete.

LEMMA 4.2. *For any $Q \geq 1$, one has*

$$\sum_{Q<q\leq 2Q} q^{-2} f(q)^2 H(q) \ll Q^{\frac{1}{k}-1+\varepsilon}.$$

The singular series $\mathfrak{S}$ defined by (4.4) converges absolutely, and one has $\mathfrak{S} = \varrho^{-1}$.

Proof. By (3.7), one has $f(q) \ll 1$, and therefore we begin with

$$\sum_{Q<q\leq 2Q} q^{-2} f(q)^2 H(q) \ll Q^{\frac{1}{k}-1} \sum_{q\leq 2Q} q^{-1-\frac{1}{k}} H(q).$$

By Lemma 4.1, we have $H(q) = 0$ unless q is $(k+1)$-free. Any $(k+1)$-free integer q has a unique representation $q = q_1 q_2^2 \cdots q_k^k$ with pairwise coprime and square-free natural numbers q_j $(1 \leq j \leq k)$. By Lemma 4.1 again, and an elementary estimate for the divisor function,

$$\sum_{q\leq 2Q} q^{-1-\frac{1}{k}} H(q) \ll Q^{\varepsilon} \sum_{q_1 q_2^2 \cdots q_k^k \leq 2Q} \prod_{\nu=1}^{k} q_\nu^{2\nu-\frac{\nu}{k}-2k} \ll Q^{2\varepsilon}$$

which confirms the first statement of the lemma. The absolute convergence of $\mathfrak{S}$ is an immediate corollary, and the general term in the series (4.4) is multiplicative as a consequence of (3.7) and Lemma 4.1. Therefore, $\mathfrak{S}$ can be rewritten as an Euler product, say

$$\mathfrak{S} = \prod_p \chi_p,$$

where by another application of (3.7) and Lemma 4.1, the Euler factor χ_p is

$$\begin{aligned} \chi_p &= 1 + \sum_{\nu=1}^{\infty} p^{-2\nu} f(p^\nu)^2 H(p^\nu) \\ &= 1 + 2\Big(1 - \frac{2}{p^k}\Big)^{-2} \left(p^{1-2k}\Big(1 - \frac{2}{p}\Big) + \sum_{\nu=2}^{k} p^{\nu-2k}\Big(1 - \frac{1}{p}\Big) \right) \\ &= \Big(1 - \frac{2}{p^k}\Big)^{-1}. \end{aligned}$$

A comparisom with (1.1) yields the identity $\mathfrak{S} = \varrho^{-1}$, as required.

5. The major arc contribution. It is time to embark on the main argument. We follow V in spirit and provide an asymptotic formula for the integral (1.8). With this end in view, let $1 \leq Q \leq \frac{1}{2}\sqrt{x}$ and $\mathfrak{M} = \mathfrak{M}(Q)$ be the set of major arcs defined prior to the statement of Theorem 2. When $|q\alpha - a| \leq Q/x$ with $1 \leq a \leq q \leq Q$ and $(a, q) = 1$ define

$$S^*(\alpha) = q^{-1} G(q, a) I\Big(\alpha - \frac{a}{q}\Big), \tag{5.1}$$

where

$$I(\beta)=\sum_{n\le x} e(\beta n)$$

and $G(q,a)$ is given by (4.1). This defines a function S^* on $\mathfrak{M}$ which serves as an approximation to $S(\alpha)$. The next lemma controls the error between S and S^* in mean square.

Lemma 5.1. *Suppose that* $1\le Q\le \frac{1}{2}\sqrt{x}$. *Then*

$$\int_{\mathfrak{M}(Q)} |S(\alpha)-S^*(\alpha)|^2\,d\alpha \ll Q^{3-\frac{2}{k}}x^{\frac{2}{k}-1+\varepsilon}+x^{\frac{4}{k+1}-1+\varepsilon}Q^2.$$

This lemma should be compared with Lemma 3.2 of V. The proof is almost identical save that the function $G(q,a)$ in (5.1) is, in the context of V, only a function of q. The slightly more general situation hardly affects the argument, and therefore we content ourselves with a few hints on the necessary changes. The definition (3.8) in V now takes the shape

$$u(n;q,a)=\begin{cases}\mu_k(n)\mu_k(n+1)e(an/q)-q^{-1}G(q,a), & \text{when } 1\le n\le x,\\ 0, & \text{otherwise.}\end{cases}$$

Then the proof of V, Lemma 3.2, still applies in the new context, and yields (compare V, (3.15))

$$\int_{\mathfrak{M}} |S(\alpha)-S^*(\alpha)|^2 d\alpha$$
$$\ll Q^{\varepsilon}\max_{1\le R\le Q}\Big(\frac{Q^2}{xR^2}\mathcal{G}(R)+\frac{Q}{x}\Upsilon_k(x,2R)+\frac{Q^2}{x^2R}\int_0^x \Upsilon_k(y,2R)dy\Big),$$

where $\Upsilon_k(x,Q)$ is the variance estimated in Lemma 3.2, and where

$$\mathcal{G}(R)=\sum_{R<q\le 2R} q^{-2}\sum_{\substack{a=1\\(a,q)=1}}^{q} |G(q,a)|^2.$$

By (4.2), (4.3) and Lemma 4.2,

$$\mathcal{G}(R)\ll R^{\frac{1}{k}-1+\varepsilon}, \tag{5.2}$$

and Lemma 5.1 follows by invoking Lemma 3.2 to bound Υ_k.

Lemma 5.2. *For* $1\le 2R\le Q\le \frac{1}{2}\sqrt{x}$, *one has*

$$\int_{\mathfrak{M}(2R)\setminus\mathfrak{M}(R)} |S^*(\alpha)|^2 d\alpha \ll xR^{\frac{1}{k}-1+\varepsilon}.$$

Proof. Note that for $|\beta|\le \frac{1}{2}$ one has

$$|I(\beta)|\ll x(1+x|\beta|)^{-1}. \tag{5.3}$$

Hence, the integral in question does not exceed

$$\ll \mathcal{G}(R) \int_{-\infty}^{\infty} x^2(1+x|\beta|)^{-2} d\beta + \sum_{q\le R} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} |G(q,a)|^2 \int_{R/(qx)}^{\infty} \beta^{-2} d\beta.$$

The conclusion of the lemma is now readily verified by recalling (5.2).

To establish Theorem 2, we integrate the identity

$$|S(\alpha)|^2 - |S^*(\alpha)|^2 = |S(\alpha)-S^*(\alpha)|^2 + 2\ \mathrm{Re}\ \overline{S^*(\alpha)}(S(\alpha)-S^*(\alpha))$$

over $\mathfrak{M}(Q)$. By Lemma 5.1 and a dyadic splitting up argument, it follows that

$$\int_{\mathfrak{M}(Q)} |S(\alpha)|^2 d\alpha - \int_{\mathfrak{M}(Q)} |S^*(\alpha)|^2 d\alpha \ll x^{\varepsilon}(Q^{3-\frac{2}{k}} x^{\frac{2}{k}-1} + x^{\frac{4}{k+1}-1} Q^2 + E) \quad (5.4)$$

where

$$E = \max_{1\le R\le Q} \int_{\mathfrak{M}(2R)\setminus\mathfrak{M}(R)} |S^*(\alpha)(S(\alpha)-S^*(\alpha))| d\alpha.$$

By Schwarz's inequality, Lemma 5.1 and Lemma 5.2,

$$\begin{aligned} E &\ll \max_{R\le Q} \big(xR^{\frac{1}{k}-1+\varepsilon}\big)^{\frac12} \big(R^{3-\frac{2}{k}} x^{\frac{2}{k}-1+\varepsilon} + x^{\frac{4}{k+1}-1+\varepsilon} R^2\big)^{\frac12} \\ &\ll x^{\frac{1}{k}+\varepsilon} Q^{1-\frac{1}{2k}} + x^{\frac{2}{k+1}+\varepsilon} Q^{\frac12+\frac{1}{2k}}. \end{aligned}$$

The second integral on the left hand side of (5.4) is evaluated by recalling (5.3). Since

$$\int_{-1/2}^{1/2} |I(\beta)|^2 d\beta = [x],$$

we deduce that

$$\int_{\mathfrak{M}(Q)} |S^*(\alpha)|^2 d\alpha = \sum_{q\le Q} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} |G(q,a)|^2 \Big([x] + O\Big(\frac{xq}{Q}\Big)\Big).$$

By (5.2) it follows that

$$\int_{\mathfrak{M}(Q)} |S^*(\alpha)|^2 d\alpha = x \sum_{q=1}^{\infty} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} |G(q,a)|^2 + O(xQ^{\frac{1}{k}-1+\varepsilon}).$$

By (4.2) and (4.4), the infinite sum on the right is $\varrho^2\mathfrak{S}$, and Lemma 4.2 yields

$$\int_{\mathfrak{M}(Q)} |S^*(\alpha)|^2 d\alpha = \varrho x + O(xQ^{\frac{1}{k}-1+\varepsilon}).$$

We substitute back into (5.4) and subtract the resulting formula from (1.4). Invoking (1.2) we then find that

$$\int_{\mathfrak{m}(Q)} |S(\alpha)|^2 d\alpha \\ \ll x^{\varepsilon}\big(xQ^{\frac{1}{k}-1} + Q^{3-\frac{2}{k}}x^{\frac{2}{k}-1} + x^{\frac{4}{k+1}-1}Q^2 + x^{\frac{1}{k}}Q^{1-\frac{1}{2k}} + x^{\frac{2}{k+1}}Q^{\frac{1}{2}+\frac{1}{2k}}\big).$$

Here the last two terms on the right hand side are always dominated by the others, and Theorem 2 follows.

6. A binary additive problem. We briefly sketch a proof of Theorem 3. It will now be useful to take $x = N$ in the previous analysis, and fix the value of Q as $Q = N^{1/5}$. Then, with $\mathfrak{m} = \mathfrak{m}(Q), \mathfrak{M} = \mathfrak{M}(Q)$, one has, by Theorem 2,

$$\int_{\mathfrak{m}} |S_r(\alpha)|^2 d\alpha \ll x^{\frac{9}{10}+\varepsilon}$$

for all $r \geq 2$. Since

$$r_{k,l}(N) = \int_0^1 S_k(\alpha)S_l(\alpha)e(-\alpha N)d\alpha$$

by orthogonality, we may conclude from Cauchy-Schwarz's inequality that

$$r_{k,l}(N) = \int_{\mathfrak{M}} S_k(\alpha)S_l(\alpha)e(-\alpha N)d\alpha + O(N^{\frac{9}{10}+\varepsilon}). \tag{6.1}$$

We now replace S_k and S_l by their approximations S_k^* and S_l^* defined in (5.1). Here it is now advisable to make the dependence on k and l explicit; we also apply this convention to the sums (4.1) by now writing $G_k(q,a), H_k(q,a)$, and similarly $f_k(q)$ instead of $f(q)$. For $1 \leq R \leq Q$, one has

$$\int_{\mathfrak{M}(2R)\setminus\mathfrak{M}(R)} |(S_k(\alpha) - S_k^*(\alpha))S_l(\alpha)|d\alpha \\ \leq \Big(\int_{\mathfrak{M}(2R)} |S_k(\alpha) - S_k^*(\alpha)|^2 d\alpha\Big)^{\frac{1}{2}} \left(\int_{\mathfrak{m}(R)} |S_l(\alpha)|^2 d\alpha\right)^{\frac{1}{2}}$$

whence by Lemma 5.1, Theorem 2 and a dyadic splitting up argument, we find that

$$\int_{\mathfrak{M}} |(S_k(\alpha) - S_k^*(\alpha))\, S_l(\alpha)|\, d\alpha \ll x^{\frac{9}{10}}. \tag{6.2}$$

Similarly, by applying Lemmata 5.1 and 5.2, one confirms the estimate

$$\int_{\mathfrak{M}} |S_k^*(\alpha)\, (S_l(\alpha) - S_l^*(\alpha))|\, d\alpha \ll x^{\frac{9}{10}}. \tag{6.3}$$

We now substitute $S_k^*(\alpha)$ for $S_k(\alpha)$ in (6.1), and control the error with (6.2). Then we substitute $S_l^*(\alpha)$ for $S_l(\alpha)$, and deduce from (6.3) that

$$\begin{aligned} r_{k,l}(N) &= \int_{\mathfrak{M}} S_k^*(\alpha)S_l^*(\alpha)e(-\alpha N)d\alpha + O(N^{\frac{9}{10}+\varepsilon}). \\ &= \sum_{q\le Q} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} G_k(q,a)G_l(q,a)e(-aN/q)J^*(q) + O(N^{\frac{9}{10}+\varepsilon}), \end{aligned}$$

where we write

$$J^*(q) = \int_{-Q/(qN)}^{Q/(qN)} I(\beta)^2 e(-\beta N)d\beta.$$

By (5.2) and Schwarz's inequality,

$$\sum_{R<q\le 2R} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} |G_k(q,a)G_l(q,a)| \ll R^{\frac{1}{2}(\frac{1}{k}+\frac{1}{l})-1+\varepsilon}, \tag{6.4}$$

and

$$J^*(q) = \int_{-1/2}^{1/2} I(\beta)^2 e(-\beta N)d\beta + O\Big(\frac{qN}{Q}\Big) = N + O\Big(\frac{qN}{Q}\Big). \tag{6.5}$$

By (6.4) and (6.5), we routinely deduce that

$$r_{k,l}(N) = N\sum_{q=1}^{\infty} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} G_k(q,a)G_l(q,a)e(-aN/q) + O(N^{\frac{9}{10}+\varepsilon}).$$

The infinite series on the right hand side converges absolutely, and by (4.2) factors as $\varrho_k\varrho_l\mathfrak{S}_{k,l}(N)$, where

$$\mathfrak{S}_{k,l}(N) = \sum_{q=1}^{\infty} q^{-2}f_k(q)f_l(q) \sum_{\substack{a=1\\(a,q)=1}}^{q} H_k(q,a)H_l(q,a)e(-aN/q). \tag{6.6}$$

This proves Theorem 3. We remark that arguments such as those used in the proof of Lemma 4.1 can be used to show that the innermost sum in (6.6) is a multiplicative function of q. Therefore, the singular series can be rewritten as an Euler product. Moreover, as in the proof of Lemma 4.1, one confirms that for $\nu > k \ge l \ge 2$ one has

$$\sum_{\substack{a=1\\p\nmid a}}^{p^\nu} H_k(p^\nu,a)H_l(p^\nu,a)e(-aN/p^\nu) = 0$$

irrespective of the value of N. Hence

$$\mathfrak{S}_{k,l}(N) = \prod_p (1+\omega_{k,l}(p)),$$

where

$$\omega_{k,l}(p) = \Big(1 - \frac{2}{p^k}\Big)^{-1}\Big(1 - \frac{2}{p^l}\Big)^{-1} \sum_{\nu=1}^{k} p^{-2\nu} \sum_{\substack{a=1 \\ p\nmid a}}^{p^\nu} H_k(p^\nu, a) H_l(p^\nu, a) e(-aN/p^\nu).$$

One can now follow the pattern laid down in the proof of Lemma 4.1 to compute the Euler factors explicitly. We spare the reader the tedious details.

Bibliography

[1] J. Brüdern, A. Granville, A. Perelli, R. C. Vaughan and T. D. Wooley. On the exponential sum over k-free numbers. *Philos. Trans. Roy. Soc. London Ser. A* 356 (1998) 739–761.

[2] L. Carlitz. On a problem in additive arithmetic II. *Quart. J. Math. Oxford* 3 (1932) 273–290.

[3] T. Estermann. On the representations of a number as the sum of two numbers not divisible by k-th powers. *J. London Math. Soc.* 6 (1931) 37–40.

[4] C. J. A. Evelyn and E. H. Linfoot, On a problem in the additive theory of numbers (second paper). *J. reine angew. Math.* 164 (1931) 131–140.

[5] D. R. Heath-Brown. The square sieve and consecutive square-free numbers. *Math. Ann.* 266 (1984) 251-259.

[6] C. Hooley. On the Barban-Davenport-Halberstam Theorem: X. Hardy-Ramanujan J. 21 (1998) 2–11.

[7] L. Mirsky. On the frequency of pairs of square-free numbers with a given difference. *Bull. Amer. Math. Soc.* 55 (1949) 936–939.

[8] A. Perelli. The L^1 norm of certain exponential sums in number theory: a survey. *Rend. Sem. Mat. Univ. Politec. Torino* 53 (1995) 405–418.

[9] K. M. Tsang. The distribution of r-tuples of square-free numbers. *Mathematika* 32 (1985) 265–275.

[10] R. C. Vaughan. *The Hardy-Littlewood Method*. 2nd edn. Cambridge University Press, 1997.

[11] R. C. Vaughan. On a variance associated with the distribution of general sequences in arithmetic progressions, I. *Philos. Trans. Roy. Soc. London Ser. A*, 356 (1998) 781–791.

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