ON VU'S THIN BASIS THEOREM IN WARING'S PROBLEM

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1. Introduction. A set of integers $\mathfrak{B}$ is said to be an *asymptotic basis of order* h if every sufficiently large natural number is the sum of h elements of $\mathfrak{B}$. Write $\mathbb{N}_0^k$ for the set of kth powers of non-negative integers. Expressed in this language, the number $G(k)$ familiar to additive number theorists may be defined as the least number g satisfying the property that the set $\mathbb{N}_0^k$ constitutes an asymptotic basis of order g. For larger exponents k, the best available upper bound for $G(k)$ due to Wooley [16] shows that

$$G(k) < k(\log k + \log\log k + 2 + O(\log\log k/\log k)). \tag{1.1}$$

Rather recently, Vu [14] has shown that whenever s is sufficiently large in terms of k, then there exists a thin asymptotic basis $\mathfrak{X}_k \subseteq \mathbb{N}_0^k$ of finite order s. When we refer to $\mathfrak{X}_k$ as being "thin", we mean that for every large number t, one has

$$\mathrm{card}(\mathfrak{X}_k \cap [1,t]) \ll (t\log t)^{1/s} \tag{1.2}$$

(see Theorem 4.1 of Vu [14]; we refer the reader to [2], [4], [7], [15] and [17] for earlier conclusions relevant to the case $k = 2$, and to [3] and [6] for weaker conclusions available previously for $k > 2$). Although Vu does not record explicitly how large s must be in order that the conclusion (1.2) be valid, a careful reading of the paper, together with a perusal of the associated references, reveals that one must take s of size somewhat larger than $2k^3\log k$, or possibly even $k2^k$, in order that Vu's argument be applicable. Indeed, Vu's main theorem establishes information on the number of representations underlying the above discussion, provided that s is very much larger in terms of k (see Theorem 1.2 of Vu [14], and Theorem 1.1 and the ensuing discussion below). In this paper we establish the existence of a thin asymptotic basis $\mathfrak{X}_k$ of finite order s, satisfying the property (1.2), whenever $s > (1+o(1))k\log k$. Indeed, we are essentially able to show that a thin asymptotic basis of order s exists whenever current technology from the Hardy-Littlewood method permits one to establish that $G(k) \leqslant s-2$.

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In order to describe our main conclusion precisely, we must introduce some notation. At this stage we concentrate on simplicity of exposition rather than precision, and we defer a more technical discussion applicable to smaller exponents k to §5 below. Let k be an integer with $k \geqslant 3$. For each natural number s, define the positive number $\Delta_s = \Delta_s(k)$ to be the unique solution of the equation

$$\Delta_s e^{\Delta_s/k} = k e^{1-2s/k}. \tag{1.3}$$

Also, define the exponent $\sigma(k)$ by putting

$$\sigma(k) = \max_{\substack{s,t,w \in \mathbb{N} \\ 2s \geqslant k+1}} \frac{k - \Delta_t - \Delta_s \Delta_w}{2(s(k + \Delta_w - \Delta_t) + tw(1 + \Delta_s))}. \tag{1.4}$$

We remark that the argument of the proof of Corollary 2 to Theorem 4.2 of Wooley [16] shows that when k is large, one has

$$\sigma(k)^{-1} = k(\log k + O(\log \log k)). \tag{1.5}$$

Next, we define $\mathfrak{G}(k)$ by taking

$$\mathfrak{G}^*(k) = \max_{v \geqslant k} \{2v + 3 + 2[\Delta_v/(2\sigma(k))]\},$$

and then putting

$$\mathfrak{G}(k) = \min\{\mathfrak{G}^*(k)\, ,\, 2^k + 1\}. \tag{1.6}$$

Finally, when $\mathfrak{X}_k \subseteq \mathbb{N}_0^k$, we write $R_s(n; \mathfrak{X}_k)$ for the number of solutions of the equation

$$x_1^k + x_2^k + \cdots + x_s^k = n,$$

with $x_i^k \in \mathfrak{X}_k$ $(1 \leqslant i \leqslant s)$.

Theorem 1.1. *Suppose that k and s are natural numbers with $k \geqslant 3$ and $s \geqslant \mathfrak{G}(k)$. Then there exists a subset $\mathfrak{X}_k = \mathfrak{X}_k(s)$ of $\mathbb{N}_0^k$ such that, when n is sufficiently large in terms of k and s, one has*

$$\log n \ll R_s(n; \mathfrak{X}_k) \ll \log n. \tag{1.7}$$

In particular, the cardinality of the set $\mathfrak{X}_k(s)$ satisfies the condition (1.2).

By computing an asymptotic expansion of $\mathfrak{G}(k)$ via the argument of §5 of Wooley [16], one derives the following immediate corollary.

Corollary 1. *For each natural number k with $k \geqslant 3$, there exists a thin asymptotic basis $\mathfrak{X}_k \subseteq \mathbb{N}_0^k$ of order*

$$\mathfrak{G}(k) = k(\log k + \log \log k + 2 + O(\log \log k / \log k)).$$

A comparison of the conclusion of this corollary with the upper bound (1.1) reveals that, for larger values of k, almost nothing is lost by restricting bases to thin sets. We note that refinements of Theorem 1.1 for smaller values of k are discussed in §5.

As remarked earlier, Vu has shown that a conclusion similar to this corollary holds in which the order of the basis $\mathfrak{X}_k$ is somewhat larger than $2k^3 \log k$ (see Theorem 4.1 of Vu [14]). The sharper conclusion recorded in Theorem 1.1, on the other hand, is made available by Vu [14] only for $s > Ck^4 8^k$, for a suitable positive constant C (see Theorem 1.2 of Vu [14], and the discussion initiating §§2.2 and 3 of that paper; here we have corrected an arithmetic oversight in the discussion at the start of §2.2 of Vu [14], wherein it is evident that one must take $s_3(k)$ as dK^3k^3 in place of dK^3k^2). In contrast to this previous work of Vu, our new conclusions are applicable as soon as the implicit number of variables is as large, essentially speaking, as the upper bound for $G(k)$ available from current technology in Waring's problem.

As is pointed out in Vu [14], a thin asymptotic basis $\mathfrak{X}_k \subseteq \mathbb{N}_0^k$ may always be converted to a thin basis $\widehat{\mathfrak{X}}_k$ of order h, in which *every* natural number is the sum of h elements of $\widehat{\mathfrak{X}}_k$, provided that h is not too small. When $k \geqslant 2$, let $g(k)$ denote the smallest number s with the property that every positive integer is the sum of at most s kth powers of natural numbers. Suppose that $\mathfrak{X}_k$ is a thin asymptotic basis of order h, and let N_0 be the largest integer that is not a sum of h elements of $\mathfrak{X}_k$. Then provided only that $h \geqslant g(k)$, it follows that the set

$$\widehat{\mathfrak{X}}_k = \mathfrak{X}_k \cup \{0, 1^k, 2^k, \dots, [N_0^{1/k}]^k\}$$

forms a thin basis of order h. We therefore obtain the following consequence of the methods underlying Theorem 1.1, and here we make use of the current knowledge concerning the value of $g(k)$ (see, for example, Chapter 1 of Vaughan [10], and associated references).

Corollary 2. *Suppose that k is an integer with $k \geqslant 3$. Then there exists a basis $\widehat{\mathfrak{X}}_k$ of order s, with*

$$\textit{card}(\widehat{\mathfrak{X}}_k \cap [1, t]) \ll (t \log t)^{1/s}$$

for each positive number t, if and only if $s \geqslant g(k)$.

It may be illuminating to recall here that it is now known that whenever $k \geqslant 2$, one has

$$g(k) = 2^k + \left[(3/2)^k\right] - 2,$$

provided that

$$2^k \left\{(3/2)^k\right\} + \left[(3/2)^k\right] \leqslant 2^k,$$

and that when this condition fails, one has

$$g(k) = 2^k + \left[(3/2)^k\right] + \left[(4/3)^k\right] - \delta,$$

where δ is 2 or 3 according to whether

$$\left[(4/3)^k\right]\left[(3/2)^k\right]+\left[(4/3)^k\right]+\left[(3/2)^k\right]$$

equals or exceeds 2^k.

As with Vu's treatment of this problem, our argument has two phases. Rather than work with the complete set of integers, as Vu does, we consider instead a special set $\mathcal{A}$ of smooth numbers that comprise a convenient subset of $\mathbb{N}$ on which to base our analysis. Such subsets permit the use of the powerful new technology involving smooth Weyl sums associated with the latest developments in Waring's problem. In the first phase of our argument, whenever $m \in \mathbb{N}$ and $s \geqslant \mathfrak{G}(k)$, we seek to establish the bounds

$$1 \ll \sum_{\substack{x_i \in \mathcal{A}\ (1\leqslant i\leqslant s)\\ x_1^k+\dots+x_s^k=m}} (x_1\dots x_s)^{-1+k/s} \ll 1, \tag{1.8}$$

and also, when $1 \leqslant l < s$, we establish the related upper bound

$$\sum_{\substack{x_i \in \mathcal{A}\ (1\leqslant i\leqslant l)\\ x_1^k+\dots+x_l^k=m}} (x_1\dots x_l)^{-1+k/s} \ll m^{-\mu}, \tag{1.9}$$

for a suitable positive number μ. Rather than proceed by dividing the ranges for the variables into many boxes, which was the approach adopted by Vu [14], we instead recognise that

$$\sum_{\substack{x_i \in \mathcal{A}\ (1\leqslant i\leqslant s)\\ x_1^k+\dots+x_s^k=m}} (x_1\dots x_s)^{-1+k/s} = \int_0^1 F(\alpha)^s e(-m\alpha)d\alpha, \tag{1.10}$$

where

$$F(\alpha)=\sum_{x\in\mathcal{A}\cap[1,m^{1/k}]} x^{-1+k/s}e(\alpha x^k),$$

and, as usual, we write $e(z)$ for $e^{2\pi iz}$. The integral (1.10) may be estimated via the Hardy-Littlewood method, the weights causing only technical difficulties. We note, however, that several manoeuvres from the repertoire of the circle method enthusiast are required in order to successfully negotiate all of the difficulties encountered.

The bounds (1.8) and (1.9) are exploited in the second phase of our analysis by means of Vu's probabilistic analysis, the only innovation in our argument being some additional input from the circle method (see the discussion at the start of §4 below, and in particular Lemmata 4.4 and 4.5). We note, however, that this minor improvement in the second phase accounts, by itself, for a factor k improvement in the upper bounds for the number of variables employed.

Throughout, the letter ε will denote a sufficiently small positive number, and P will be a large real number. We use $\ll$ and $\gg$ to denote Vinogradov's notation, and write $f \asymp g$ when $g \ll f$ and $f \ll g$. We write $\lceil \alpha \rceil$ for the smallest integer at least as large as α, and $[\alpha]$ for the greatest integer not exceeding α. We then put $\{\alpha\} = \alpha - [\alpha]$. Also, we write $\|\alpha\|$ for $\min_{y\in\mathbb{Z}} |\alpha - y|$. Finally, in an effort to simplify our account, whenever ε appears in a statement, we assert that the statement holds for every positive number ε. Thus the "value" of ε may change from statement to statement.

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2. Auxiliary mean value estimates. Our application of the Hardy-Littlewood method involves the use of some mean value estimates for modified smooth Weyl sums. Before describing these estimates, we require some notation. We fix a natural number k with $k \geqslant 3$, define $g = \mathfrak{G}(k)$ by means of the formulae (1.3)–(1.6), and put $\theta = 1 + 10^{-100}k^{-2}$. When $1 \leqslant R \leqslant P$, we define the set $\mathcal{A}^*(P,R)$ of R-smooth numbers up to P by

$$\mathcal{A}^*(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p \text{ prime and } p|n \Rightarrow \sqrt{R} < p \leqslant R\},$$

and we write $\mathcal{B}(P,R)$ for $\mathcal{A}^*(P,R) \setminus \mathcal{A}^*(\theta^{-1}P,R)$. We then define the associated set $\mathcal{C}(P,R)$ by

$$\mathcal{C}(P,R) = \bigcup_{h=0}^{H} \{lm : m \in \mathcal{B}(\theta^h P/\sqrt{R}, R) \text{ and } \theta^{-h-1}\sqrt{R} < l \leqslant \theta^{-h}\sqrt{R}\}, \tag{2.1}$$

where

$$H = \left[\frac{\log R}{8\log\theta}\right].$$

We remark that whenever $n \in \mathcal{C}(P,R)$, then n is uniquely represented in the shape $n = lm$ with

$$m \in \mathcal{B}(\theta^h P/\sqrt{R}, R), \quad \theta^{-h-1}\sqrt{R} < l \leqslant \theta^{-h}\sqrt{R} \quad \text{and} \quad 0 \leqslant h \leqslant H,$$

as is apparent by considering the prime factorisation of n. Observe also that whenever $n \in \mathcal{C}(P,R)$, then necessarily $n \in (\theta^{-2}P, P]$. Finally, when η is a real number with $0 < \eta \leqslant 1$, we define the infinite set $\mathcal{A}_\eta$ by

$$\mathcal{A}_\eta = \bigcup_{m=1}^{\infty} \mathcal{C}(\theta^{2m}, \theta^{2m\eta}), \tag{2.2}$$

and we write $\mathcal{A}_\eta(P)$ for $\mathcal{A}_\eta \cap [1,P]$. Note that if we define the conventional set of R-smooth numbers not exceeding P by

$$\mathcal{A}(P,R) = \{n \in [1,P] \cap \mathbb{Z} : p \text{ prime and } p|n \Rightarrow p \leqslant R\},$$

then one plainly has $\mathcal{A}_\eta(P) \subseteq \mathcal{A}(P, P^\eta)$.

We must next define the exponential sums that are the key characters in our argument. When t is a natural number and P is a large positive number, we write

$$M_+ = \left\lceil \frac{\log P}{\log(\theta^2)} \right\rceil, \quad M_- = \left[\frac{\log(t^{-1/k}P)}{\log(\theta^2)} \right] - 1,$$

and then define

$$P_+ = \theta^{2M_+} \quad \text{and} \quad P_- = \theta^{2M_-}.$$

Note that

$$P \leqslant P_+ < \theta^2 P \quad \text{and} \quad \theta^{-4} t^{-1/k} P < P_- \leqslant \theta^{-2} t^{-1/k} P.$$

For later use, it is convenient also to define

$$H_\eta(m) = [m\eta/4].$$

When η is a real number with $0 < \eta \leqslant 1$, we define the exponential sums $f(\alpha) = f_{t,\eta}(\alpha; P)$ and $h(\alpha) = h_{t,\eta}(\alpha; P)$ by

$$f_{t,\eta}(\alpha; P) = \sum_{x \in \mathcal{A}_\eta(P_+)} x^{-1+k/t} e(\alpha x^k) \tag{2.3}$$

and

$$h_{t,\eta}(\alpha; P) = \sum_{\substack{x \in \mathcal{A}_\eta(P_+) \\ x > P_-}} x^{-1+k/t} e(\alpha x^k). \tag{2.4}$$

Our first mean value estimate provides a crude, yet effective, bound for the moments of the sum $f(\alpha)$.

Lemma 2.1. *Suppose that t and w are natural numbers with $t \geqslant w \geqslant \mathfrak{G}(k) - 1$. Suppose also that η is a positive number sufficiently small in terms of k and w, and that P is a positive number sufficiently large in terms of k, w and η. Then one has*

$$\int_0^1 |f_{t,\eta}(\alpha; P)|^w d\alpha \ll (\log P)^w.$$

Here, the implicit constant in Vinogradov's notation may depend at most on t, k and η.

Proof. Define the exponential sums

$$g_{t,\eta}(\alpha; Q) = \sum_{x \in \mathcal{C}(Q, Q^\eta)} x^{-1+k/t} e(\alpha x^k),$$

$$\mathfrak{h}_\eta(\alpha; Q) = \sum_{x \in \mathcal{A}(Q, Q^\eta)} e(\alpha x^k) \quad \text{and} \quad H(\alpha; Q) = \sum_{\theta^{-2} Q < x \leqslant Q} e(\alpha x^k).$$

Then we find from (2.2) and (2.3) that

$$f_{t,\eta}(\alpha;P)=\sum_{m=1}^{M_+} g_{t,\eta}(\alpha;\theta^{2m}).$$

Write $u=[w/2]$, and put $\nu=w-2u$. Observe that a trivial estimate for $g_{t,\eta}(\alpha;Q)$ leads to the upper bound

$$|g_{t,\eta}(\alpha;Q)|\leqslant \sum_{1\leqslant x\leqslant Q} x^{-1+k/t}\ll Q^{k/t}.$$

Then on combining this trivial estimate for $g_{t,\eta}(\alpha;Q)$ with Hölder's inequality, one obtains

$$\int_0^1 |f_{t,\eta}(\alpha;P)|^w d\alpha \ll M_+^{w-1}\sum_{m=1}^{M_+}(\theta^{2mk/t})^\nu \int_0^1 |g_{t,\eta}(\alpha;\theta^{2m})|^{2u}d\alpha. \tag{2.5}$$

Observe next that, from orthogonality, it follows that the mean value

$$\int_0^1 |g_{t,\eta}(\alpha;Q)|^{2u}d\alpha$$

is equal to the number of integral solutions of the equation

$$x_1^k+\dots+x_u^k=x_{u+1}^k+\dots+x_{2u}^k,$$

with $x_i\in\mathcal{C}(Q,Q^\eta)$ $(1\leqslant i\leqslant 2u)$, and with each solution $\mathbf{x}$ being counted with weight

$$(x_1x_2\dots x_{2u})^{-1+k/t}\ll (Q^{2u})^{-1+k/t}.$$

On recalling that $\mathcal{C}(Q,Q^\eta)\subseteq\mathcal{A}(Q,Q^\eta)$, and also that $\mathcal{C}(Q,Q^\eta)\subseteq(\theta^{-2}Q,Q]$, we deduce from orthogonality that

$$\int_0^1 |g_{t,\eta}(\alpha;Q)|^{2u}d\alpha \ll (Q^{2u})^{-1+k/t}\int_0^1 |H(\alpha;Q)^2\mathfrak{h}_\eta(\alpha;Q)^{2u-2}|d\alpha. \tag{2.6}$$

However, the argument of §5 of Wooley [16] (see also §8 of Vaughan and Wooley [13]) shows that the mean value on the right hand side of (2.6) is $O(Q^{2u-k})$ whenever $2u\geqslant\mathfrak{G}(k)-1$. Then the upper bound (2.6) yields the relatively sharp estimate

$$\int_0^1 |g_{t,\eta}(\alpha;Q)|^{2u}d\alpha\ll Q^{k(2u/t-1)}. \tag{2.7}$$

Finally, on substituting (2.7) into (2.5), we find that whenever $t\geqslant w\geqslant\mathfrak{G}(k)-1$, one has

$$\begin{aligned}\int_0^1 |f_{t,\eta}(\alpha;P)|^w d\alpha &\ll (\log P)^w \max_{Q\leqslant P_+}(Q^{k/t})^\nu(Q^{k(2u/t-1)})\\ &\ll (\log P)^w\max_{Q\leqslant P_+}Q^{k(w/t-1)}\ll(\log P)^w.\end{aligned}$$

Here, the implicit constant in Vinogradov's notation depends at most on w, k and η, and thus the conclusion of the lemma follows whenever P is sufficiently large in terms of the latter quantities.

We require also an analogue of Lemma 2.1 of use for the exponential sum $h_{t,\eta}(\alpha;P)$, though here we are able to derive a sharper conclusion.

Lemma 2.2. *Suppose that t and w are natural numbers with $t \geqslant w \geqslant \mathfrak{G}(k) - 1$. Suppose also that η is a positive number sufficiently small in terms of k, and that P is a positive number sufficiently large in terms of w, k and η. Then one has*

$$\int_0^1 |h_{t,\eta}(\alpha; P)|^w d\alpha \ll P^{k(w/t-1)}.$$

Here, the implicit constant in Vinogradov's notation may depend at most on t, k and η.

Proof. We find from (2.2) and (2.4) that

$$h_{t,\eta}(\alpha; P) = \sum_{m=M_-+1}^{M_+} g_{t,\eta}(\alpha; \theta^{2m}),$$

where the exponential sum $g_{t,\eta}(\alpha; Q)$ is defined as in the proof of Lemma 2.1. Again, we write $u = [w/2]$, and put $\nu = w - 2u$. Then by combining our trivial estimate for $g_{t,\eta}(\alpha; Q)$ with Hölder's inequality, we obtain

$$\int_0^1 |h_{t,\eta}(\alpha; P)|^w d\alpha \ll (M_+ - M_-)^{w-1} \sum_{m=M_-+1}^{M_+} (\theta^{2mk/t})^\nu \int_0^1 |g_{t,\eta}(\alpha; \theta^{2m})|^{2u} d\alpha. \tag{2.8}$$

On noting that $M_+ - M_- \ll_{t,k} 1$, and recalling the estimate (2.7) from the proof of Lemma 2.1, we conclude from (2.8) that

$$\int_0^1 |h_{t,\eta}(\alpha; P)|^w d\alpha \ll \max_{\theta^{-4}t^{-1/k}P < Q \leqslant \theta^2 P} (Q^{k/t})^\nu (Q^{k(2u/t-1)}) \ll P^{k(w/t-1)}.$$

Here, the constants implicit in Vinogradov's notation may depend at most on t, k and η. This completes the proof of the lemma.

Finally, we provide a mean value estimate that yields useful information when one of the underlying variables is relatively small.

Lemma 2.3. *Suppose that t is a natural number with $t \geqslant \mathfrak{G}(k)$. Suppose also that η is a positive number sufficiently small in terms of k, and that P is a positive number sufficiently large in terms of t, k and η. Then whenever $Q \leqslant P^{1/t}$ and $1 \leqslant v \leqslant t-1$, one has*

$$\int_0^1 |h_{t,\eta}(\alpha; P) f_{t,\eta}(\alpha; P)^{t-v-1} f_{t,\eta}(\alpha; Q)^v| d\alpha \ll P^{-k/t^3}.$$

Proof. A trivial estimate for $f_{t,\eta}(\alpha; Q)$ yields the bound

$$|f_{t,\eta}(\alpha; Q)| \ll Q^{k/t} \ll P^{k/t^2}.$$

It therefore follows from Hölder's inequality that

$$\int_0^1 |h_{t,\eta}(\alpha;P) f_{t,\eta}(\alpha;P)^{t-v-1} f_{t,\eta}(\alpha;Q)^v| d\alpha \\ \ll P^{k/t^2} I_1^{1-v/(t-1)} I_2^{(v-1)/(t-1)} I_3^{1/(t-1)},$$

where

$$I_1 = \int_0^1 |f_{t,\eta}(\alpha;P)|^{t-1} d\alpha, \quad I_2 = \int_0^1 |f_{t,\eta}(\alpha;Q)|^{t-1} d\alpha$$

and

$$I_3 = \int_0^1 |h_{t,\eta}(\alpha;P)|^{t-1} d\alpha.$$

But Lemma 2.1 provides the upper bound $I_i \ll (\log P)^t$ $(i = 1, 2)$, and Lemma 2.2 shows that $I_3 \ll P^{-k/t}$. Thus we deduce that

$$\int_0^1 |h_{t,\eta}(\alpha;P) f_{t,\eta}(\alpha;P)^{t-v-1} f_{t,\eta}(\alpha;Q)^v| d\alpha \ll (\log P)^t P^{-k/(t^2(t-1))},$$

and the conclusion of the lemma is immediate whenever P is sufficiently large in terms of t, k and η.

3. Application of the Hardy-Littlewood method. In this section we seek to obtain good upper and lower bounds for the weighted sum

$$Y_{s,\eta}(n) = \sum_{\substack{x_1,\dots,x_s \in \mathcal{A}_\eta \\ x_1^k + \dots + x_s^k = n}} (x_1 \dots x_s)^{-1+k/s},$$

valid for appropriate ranges of s and η. This we achieve by means of the Hardy-Littlewood method, and it is this that constitutes the most difficult aspect of this paper. We summarise the conclusion of this section as the following theorem.

Theorem 3.1. *Suppose that $k \geqslant 3$, that $s \geqslant \mathfrak{G}(k)$, and that η is a positive number sufficiently small in terms of k. Then whenever n is sufficiently large in terms of s, k and η, there exist positive numbers $\Xi_\pm = \Xi_\pm(s,k,\eta)$, independent of n, with the property that*

$$\Xi_-(s,k,\eta) \leqslant Y_{s,\eta}(n) \leqslant \Xi_+(s,k,\eta).$$

We remark that with some additional effort, it is possible to obtain an asymptotic formula for $Y_{s,\eta}(n)$.

In order to initialise our application of the circle method, we put $P = n^{1/k}$, and recall the definitions of $P_\pm$ from §2, wherein we set $t = s$. It follows that whenever

$$x_1^k + \dots + x_s^k = n,$$

with $x_i \in \mathbb{N}$, then necessarily

$$\max_{1\leqslant i\leqslant s} x_i \geqslant (n/s)^{1/k} > P_-.$$

On recalling the definitions (2.3) and (2.4), it is therefore apparent from orthogonality that

$$Y_{s,\eta}(n) \asymp Y^+_{s,\eta}(n), \tag{3.1}$$

where

$$Y^+_{s,\eta}(n) = \int_0^1 h_{s,\eta}(\alpha;P) f_{s,\eta}(\alpha;P)^{s-1} e(-n\alpha)\,d\alpha. \tag{3.2}$$

Before proceeding further, we dispose of solutions implicit in (3.2) in which one or more variables are unusually small. In this context, we write

$$M_s = \left[\frac{\log(n^{1/(ks)})}{\log(\theta^2)}\right] - 1, \quad P_s = \theta^{2M_s},$$

and define

$$\widehat{f}_{s,\eta}(\alpha;P) = f_{s,\eta}(\alpha;P) - f_{s,\eta}(\alpha;P_s). \tag{3.3}$$

We then put

$$\widehat{Y}_{s,\eta}(n) = \int_0^1 h_{s,\eta}(\alpha;P) \widehat{f}_{s,\eta}(\alpha;P)^{s-1} e(-n\alpha)\,d\alpha. \tag{3.4}$$

The parameters s, η and P may at this point be considered fixed, with $s \geqslant \mathfrak{G}(k)$, with $\eta > 0$ sufficiently small in terms of k, and with P sufficiently large in terms of s, k and η.

Lemma 3.2. *One has*

$$Y^+_{s,\eta}(n) - \widehat{Y}_{s,\eta}(n) \ll n^{-1/s^3}.$$

Proof. One has

$$|f_{s,\eta}(\alpha;P)^{s-1} - \widehat{f}_{s,\eta}(\alpha;P)^{s-1}| \ll \sum_{v=1}^{s-1} |f_{s,\eta}(\alpha;P)^{s-1-v} f_{s,\eta}(\alpha;P_s)^v|,$$

and thus it follows from (3.2) and (3.4) via Lemma 2.3 that

$$\begin{aligned} Y^+_{s,\eta}(n) - \widehat{Y}_{s,\eta}(n) &\ll \sum_{v=1}^{s-1} \int_0^1 |h_{s,\eta}(\alpha;P) f_{s,\eta}(\alpha;P)^{s-1-v} f_{s,\eta}(\alpha;P_s)^v|\,d\alpha \\ &\ll P^{-k/s^3}. \end{aligned}$$

The conclusion of the lemma follows on recalling that $P = O(n^{1/k})$.

Next write $L = \exp(\log n/\sqrt{\log\log n})$, and define the set of major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |\alpha - a/q| \leqslant LP^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant L$ and $(a,q) = 1$. We then denote the corresponding set of minor arcs by $\mathfrak{m} = [0,1) \setminus \mathfrak{M}$. When $\mathfrak{B} \subseteq [0,1)$, it is convenient to define

$$\widehat{Y}_{s,\eta}(n;\mathfrak{B}) = \int_{\mathfrak{B}} h_{s,\eta}(\alpha;P)\widehat{f}_{s,\eta}(\alpha;P)^{s-1}e(-n\alpha)d\alpha, \tag{3.5}$$

and it is then evident that

$$\widehat{Y}_{s,\eta}(n) = \widehat{Y}_{s,\eta}(n;\mathfrak{m}) + \widehat{Y}_{s,\eta}(n;\mathfrak{M}). \tag{3.6}$$

We first analyse the contribution of the minor arcs to $\widehat{Y}_{s,\eta}(n)$, beginning with an analogue of Weyl's inequality. We save space here by establishing a rather weak bound that nonetheless suffices for our purposes.

Lemma 3.3. *One has*

$$\sup_{\alpha\in\mathfrak{m}} |h_{s,\eta}(\alpha;P)| \ll P^{k/s}L^{-2^{-2k}}.$$

Proof. We proceed in some generality. Let U and V be large positive numbers, and suppose that $\mathcal{U} \subseteq (\theta^{-1}U, U] \cap \mathbb{Z}$. It is convenient to write $Z = UV$. Let δ be a real number with $0 < \delta < 1$, and define the exponential sum $\Upsilon(\alpha) = \Upsilon(\alpha;U,V)$ by

$$\Upsilon(\alpha;U,V) = \sum_{u\in\mathcal{U}} \sum_{\theta^{-1}V<v\leqslant V} (uv)^{-\delta}e(\alpha(uv)^k). \tag{3.7}$$

The exponential sum $h_{s,\eta}(\alpha;P)$ may be decomposed as a sum of relatively few exponential sums of the shape $\Upsilon(\alpha;U,V)$, and so a reasonable version of Weyl's inequality for the latter suffices to establish the conclusion of the lemma.

Write

$$\Phi(\gamma) = \sum_{\theta^{-1}V<v\leqslant V} v^{-\delta}e(\gamma v^k) \quad \text{and} \quad \Psi(\gamma) = \sum_{\theta^{-1}U<u\leqslant U} e(\gamma u^k). \tag{3.8}$$

Also, define the polynomial $p(t;\mathbf{w})$ by

$$p(t;\mathbf{w}) = \tfrac{1}{2}k!w_1\dots w_{k-1}(2t + w_1 + \cdots + w_{k-1}). \tag{3.9}$$

Then by a simple variant of the familiar Weyl differencing lemma (see, for example, Lemma 2.3 of Vaughan [10]), one finds that

$$|\Phi(\gamma)|^{2^{k-1}} \leqslant (2V)^{2^{k-1}-k} \sum_{|v_1|<V} \cdots \sum_{|v_{k-1}|<V} \sum_{x\in I} \omega(x;\mathbf{v})e(p(x;\mathbf{v})\gamma),$$

in which $\omega(x;\mathbf{v})$ denotes a weight function arising from the factor $v^{-\delta}$ in (3.8), and one that satisfies
$$|\omega(x;\mathbf{v})| \ll (V^{-\delta})^{2^{k-1}}, \tag{3.10}$$
and $I = I(\mathbf{v})$ is an interval of integers contained in $(\theta^{-1}V, V]$. An application of Hölder's inequality to (3.7) therefore reveals that
$$\begin{aligned}|\Upsilon(\alpha)|^{2^{k-1}} &\ll U^{2^{k-1}-1} \sum_{\theta^{-1}U<u\leqslant U} |u^{-\delta}\Phi(\alpha u^k)|^{2^{k-1}}\\ &\ll U^{-1+(1-\delta)2^{k-1}} V^{2^{k-1}-k} \sum_{x,\mathbf{v}} |\omega(x;\mathbf{v})\Psi(\alpha p(x;\mathbf{v}))|,\end{aligned} \tag{3.11}$$
where the summation is over integers x, $\mathbf{v}$ satisfying
$$|v_i| < V \quad (1 \leqslant i \leqslant k-1) \quad \text{and} \quad \theta^{-1}V < x \leqslant V. \tag{3.12}$$

On recalling the upper bound (3.10), we find that an application of Hölder's inequality to (3.11) leads to the estimate
$$|\Upsilon(\alpha)|^{2^{2k-2}} \ll \left(U^{-1+(1-\delta)2^{k-1}} V^{-k+(1-\delta)2^{k-1}}\right)^{2^{k-1}} V^{k(2^{k-1}-1)}\Omega, \tag{3.13}$$
where
$$\Omega = \sum_{x,\mathbf{v}} |\Psi(\alpha p(x;\mathbf{v}))|^{2^{k-1}},$$
and the summation over x and $\mathbf{v}$ again satisfies (3.12). A second application of the Weyl differencing lemma, however, shows that
$$|\Psi(\gamma)|^{2^{k-1}} \leqslant (2U)^{2^{k-1}-k} \sum_{|u_1|<U} \cdots \sum_{|u_{k-1}|<U} \sum_{y\in J} e(p(y;\mathbf{u})\gamma),$$
where $J = J(\mathbf{u})$ is an interval of integers contained in $(\theta^{-1}U, U]$. On substituting the latter bound into (3.13), we arrive at the estimate
$$|\Upsilon(\alpha)|^{2^{2k-2}} \ll Z^{(1-\delta)2^{2k-2}-k} \sum_{x,\mathbf{v}} \sum_{y,\mathbf{u}} e(\alpha p(x;\mathbf{v})p(y;\mathbf{u})), \tag{3.14}$$
where the summation is over integers $x, \mathbf{v}$ and $y, \mathbf{u}$ satisfying (3.12) and
$$|u_i| < U \quad (1 \leqslant i \leqslant k-1) \quad \text{and} \quad \theta^{-1}U < y \leqslant U. \tag{3.15}$$

On recalling (3.9), it is evident that
$$\sum_{\theta^{-1}V<x\leqslant V} e(\alpha p(x;\mathbf{v})p(y;\mathbf{u})) \ll \min\{V, \|k! v_1 \dots v_{k-1} p(y;\mathbf{u})\alpha\|^{-1}\}, \tag{3.16}$$

where we interpret $\min\{V, \|0\|^{-1}\}$ to be V. But the expression

$$k!v_1 \dots v_{k-1} p(y; \mathbf{u})$$

plainly vanishes for at most $O(U^{k-1}V^{k-2}(U+V))$ values of $y, \mathbf{u}, \mathbf{v}$ satisfying (3.12) and (3.15). Write

$$\Xi = k!(k+1)!U^k V^{k-1} \quad \text{and} \quad \mathcal{L} = \exp(\log \Xi / \sqrt{\log\log \Xi}),$$

and observe that for each positive number ε, whenever $1 \leqslant \xi \leqslant \Xi$, the number of divisors of ξ is $O(\mathcal{L}^\varepsilon)$. Then we find from substituting (3.16) into (3.14) that

$$|\Upsilon(\alpha)|^{2^{2k-2}} \ll Z^{(1-\delta)2^{2k-2}-k}(Z^k(U^{-1}+V^{-1}) + \mathcal{L}^\varepsilon \Theta), \tag{3.17}$$

where

$$\Theta = \sum_{1 \leqslant \xi \leqslant \Xi} \min\{Z^k/\xi, \|\alpha\xi\|^{-1}\}.$$

Suppose that $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ satisfy the property that $(a,q) = 1$ and $|\alpha - a/q| \leqslant q^{-2}$. Then it follows from Lemma 2.2 of Vaughan [10] that

$$\Theta \ll Z^k(q^{-1} + V^{-1} + qZ^{-k}) \log(2qZ),$$

whence by (3.17),

$$|\Upsilon(\alpha)|^{2^{2k-2}} \ll \mathcal{L}^\varepsilon Z^{(1-\delta)2^{2k-2}}(q^{-1} + U^{-1} + V^{-1} + qZ^{-k}) \log(2qZ).$$

On noting that the latter estimate is trivial for $q > Z^k$, we conclude at last that

$$|\Upsilon(\alpha)| \ll \mathcal{L}^\varepsilon Z^{1-\delta}(q^{-1} + U^{-1} + V^{-1} + qZ^{-k})^{2^{2-2k}}. \tag{3.18}$$

Suppose next that $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \leqslant a \leqslant q \leqslant L^{-1}P^k$, $(a,q) = 1$ and $|\alpha - a/q| \leqslant q^{-1}LP^{-k}$. But since $\alpha \in \mathfrak{m}$, one necessarily has $q > L$. On noting that the conditions here on a and q ensure that $|\alpha - a/q| \leqslant q^{-2}$, we may conclude from (3.18) that whenever $M_- < m \leqslant M_+$ and $0 \leqslant h \leqslant H_\eta(m)$, then one has

$$\begin{aligned}\sum_{u \in \mathcal{B}(\theta^{2m+h-m\eta}, \theta^{2m\eta})} \sum_{\theta^{m\eta-h-1} < v \leqslant \theta^{m\eta-h}} & (uv)^{-1+k/s} e(\alpha(uv)^k) \\ & \ll L^\varepsilon P^{k/s}(q^{-1} + P^{-\eta/5} + qP^{-k})^{2^{2-2k}} \\ & \ll P^{k/s} L^{-2^{1-2k}}.\end{aligned}$$

Then it follows from (2.1) and (2.4) that

$$\sup_{\alpha \in \mathfrak{m}} |h_{s,\eta}(\alpha; P)| \ll M_+(H_\eta(M_+) + 1) P^{k/s} L^{-2^{1-2k}} \ll P^{k/s} L^{-2^{-2k}}.$$

This completes the proof of the lemma.

In combination with Lemmata 2.1 and 2.2, the upper bound for $h_{s,\eta}(\alpha; P)$ provided by Lemma 3.3 leads easily to an acceptable estimate for $\widehat{Y}_{s,\eta}(n; \mathfrak{m})$.

Lemma 3.4. *Under the hypotheses on s, η and P described in the preamble to Lemma 3.2, one has*

$$\widehat{Y}_{s,\eta}(n;\mathfrak{m}) \ll L^{-2^{-3k}/s}.$$

Proof. An application of Hölder's inequality to (3.5) reveals that

$$\widehat{Y}_{s,\eta}(n;\mathfrak{m}) \leqslant I_4^{1/s} I_5^{1-1/s}, \tag{3.19}$$

where

$$I_4 = \int_{\mathfrak{m}} |h_{s,\eta}(\alpha;P)|^s d\alpha \quad \text{and} \quad I_5 = \int_0^1 |\widehat{f}_{s,\eta}(\alpha;P)|^s d\alpha.$$

But it follows from (3.3) and Lemma 2.1 that

$$I_5 \ll \int_0^1 |f_{s,\eta}(\alpha;P)|^s d\alpha + \int_0^1 |f_{s,\eta}(\alpha;P_s)|^s d\alpha \ll (\log P)^s. \tag{3.20}$$

Also, on combining the conclusions of Lemmata 2.2 and 3.3, one finds that

$$\begin{aligned} I_4 &\leqslant \Big(\sup_{\alpha\in\mathfrak{m}} |h_{s,\eta}(\alpha;P)| \Big) \int_0^1 |h_{s,\eta}(\alpha;P)|^{s-1} d\alpha \\ &\ll (P^{k/s} L^{-2^{-2k}})(P^{-k/s}) \ll L^{-2^{-2k}}. \end{aligned} \tag{3.21}$$

Then on substituting (3.20) and (3.21) into (3.19), we conclude that

$$\widehat{Y}_{s,\eta}(n;\mathfrak{m}) \ll (\log P)^s L^{-2^{-2k}/s} \ll L^{-2^{-3k}/s}.$$

This completes the proof of the lemma.

Before embarking on an analysis of the major arcs, we provide an auxiliary estimate required to replace the functions h and f by suitable major arc approximations. In this context, when $a\in\mathbb{Z}$, $q\in\mathbb{N}$ and $\beta\in\mathbb{R}$, we introduce the functions

$$S(q,a) = \sum_{r=1}^q e(ar^k/q) \quad \text{and} \quad v_\delta(\beta;W) = \int_{\theta^{-1}W}^W \gamma^{-\delta} e(\beta\gamma^k) d\gamma. \tag{3.22}$$

Lemma 3.5. *Let δ be a real number with $0<\delta<1$, and write*

$$F_\delta(\alpha;W) = \sum_{\theta^{-1}W < w \leqslant W} w^{-\delta} e(\alpha w^k).$$

Suppose that $\alpha\in\mathbb{R}$, $a\in\mathbb{Z}$ and $q\in\mathbb{N}$. Then one has

$$F_\delta(\alpha;W) - q^{-1}S(q,a)v_\delta(\alpha - a/q;W) \ll W^{-\delta}(q + W^k|q\alpha - a|).$$

Proof. For the sake of concision, we write $\beta = \alpha - a/q$. Then by sorting the summation into arithmetic progressions modulo q, one obtains

$$\begin{aligned}\sum_{\theta^{-1}W<w\leqslant W} & w^{-\delta}e(\alpha w^k) \\ &= \sum_{r=1}^{q} \sum_{(\theta^{-1}W-r)/q<y\leqslant(W-r)/q} (yq+r)^{-\delta}e((\beta+a/q)(yq+r)^k) \\ &= \sum_{r=1}^{q} e(ar^k/q) \sum_{(\theta^{-1}W-r)/q<y\leqslant(W-r)/q} (yq+r)^{-\delta}e(\beta(yq+r)^k). \end{aligned} \tag{3.23}$$

We replace the last sum in (3.23) by a smooth integral, replacing each term by an integral over a unit interval, with an appropriately bounded error term. For any suitably smooth functions $G(z)$ and $H(z)$, one has

$$\begin{aligned}\Big|G(z)e(H(z)) - & \int_{-1/2}^{1/2} G(z+\sigma)e(H(z+\sigma))d\sigma\Big| \\ &\leqslant \sup_{-1/2\leqslant\sigma\leqslant1/2} |G(z+\sigma)e(H(z+\sigma)) - G(z)e(H(z))|,\end{aligned}$$

so that by the Mean Value Theorem,

$$\begin{aligned}G(z)e(H(z)) - & \int_{-1/2}^{1/2} G(z+\sigma)e(H(z+\sigma))d\sigma \\ &\ll \sup_{-1/2\leqslant\sigma\leqslant1/2} \left(|G'(z+\sigma)| + |G(z+\sigma)H'(z+\sigma)|\right).\end{aligned}$$

We therefore find that the expression

$$\begin{aligned}\sum_{(\theta^{-1}W-r)/q<y\leqslant(W-r)/q} & (yq+r)^{-\delta}e(\beta(yq+r)^k) \\ & - \int_{(\theta^{-1}W-r)/q}^{(W-r)/q} (zq+r)^{-\delta}e(\beta(zq+r)^k)dz\end{aligned} \tag{3.24}$$

is

$$\ll (\theta^{-1}W)^{-\delta} + q^{-1}W\mathcal{M},$$

where

$$\mathcal{M} = \sup_{\theta^{-1}W/q\leqslant z\leqslant W/q} \left(q(zq+r)^{-1-\delta} + (zq+r)^{-\delta}|\beta q(zq+r)^{k-1}|\right).$$

But by a change of variable, the integral occurring in equation (3.24) is equal to

$$q^{-1}\int_{\theta^{-1}W}^{W} \gamma^{-\delta}e(\beta\gamma^k)d\gamma = q^{-1}v_\delta(\beta; W),$$

and hence we conclude from (3.23) that

$$\begin{aligned}&F_\delta(\alpha;W)-q^{-1}S(q,a)v_\delta(\beta;W)\\&\qquad\ll\sum_{r=1}^{q}\bigl(W^{-\delta}+q^{-1}W(qW^{-1-\delta}+W^{-\delta}|\beta qW^{k-1}|\bigr)\\&\qquad\ll qW^{-\delta}(1+W^k|\beta|).\end{aligned}$$

This completes the proof of the lemma.

Next define the functions $\widetilde{v}(\beta)$ and $\widetilde{v}_1(\beta)$ for $\beta\in\mathbb{R}$ by

$$\widetilde{v}(\beta)=\sum_{m=M_s+1}^{M_+}\sum_{h=0}^{H_\eta(m)}\sum_{u\in\mathcal{B}(\theta^{2m+h-m\eta},\theta^{2m\eta})}u^{-1+k/s}v_{1-k/s}(u^k\beta;\theta^{m\eta-h})\tag{3.25}$$

and

$$\widetilde{v}_1(\beta)=\sum_{m=M_-+1}^{M_+}\sum_{h=0}^{H_\eta(m)}\sum_{u\in\mathcal{B}(\theta^{2m+h-m\eta},\theta^{2m\eta})}u^{-1+k/s}v_{1-k/s}(u^k\beta;\theta^{m\eta-h}).\tag{3.26}$$

We may now introduce the major arc approximations $f^*(\alpha)$ and $h^*(\alpha)$, to $\widehat{f}_{s,\eta}(\alpha;P)$ and $h_{s,\eta}(\alpha;P)$, respectively. Define the functions $f^*(\alpha)$ and $h^*(\alpha)$ for $\alpha\in[0,1)$ by putting

$$f^*(\alpha)=q^{-1}S(q,a)\widetilde{v}(\alpha-a/q)\quad\text{and}\quad h^*(\alpha)=q^{-1}S(q,a)\widetilde{v}_1(\alpha-a/q),$$

when $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$, and by taking $f^*(\alpha)=0$ and $h^*(\alpha)=0$ otherwise. A fairly immediate consequence of Lemma 3.5 provides useful bounds for the quality of the approximation of $f^*(\alpha)$ to $\widehat{f}_{s,\eta}(\alpha;P)$, and $h^*(\alpha)$ to $h_{s,\eta}(\alpha;P)$, when $\alpha\in\mathfrak{M}$.

Lemma 3.6. *Uniformly for $\alpha\in\mathfrak{M}$, one has*

$$\widehat{f}_{s,\eta}(\alpha;P)-f^*(\alpha)\ll P^{k/s}L^{-4}$$

and

$$h_{s,\eta}(\alpha;P)-h^*(\alpha)\ll P^{k/s}L^{-4}.$$

Proof. Making use of the notation introduced in the statement of Lemma 3.5, it is apparent from (2.3) and (3.3) that

$$\widehat{f}_{s,\eta}(\alpha;P)=\sum_{m=M_s+1}^{M_+}\sum_{h=0}^{H_\eta(m)}\sum_{u\in\mathcal{B}(\theta^{2m+h-m\eta},\theta^{2m\eta})}u^{-1+k/s}F_{1-k/s}(\alpha u^k;\theta^{m\eta-h}).\tag{3.27}$$

But from Lemma 3.5 it follows that whenever $\alpha \in \mathfrak{M}(q,a) \subseteq \mathfrak{M}$ and $u \in \mathbb{N}$, one has

$$\begin{aligned} F_{1-k/s}(\alpha u^k;\theta^{m\eta-h}) - q^{-1}S(q,a)v_{1-k/s}(u^k(\alpha-a/q);\theta^{m\eta-h}) \\ \ll q(\theta^{m\eta-h})^{-1+k/s}(1+(\theta^{m\eta-h}u)^k|\alpha-a/q|) \\ \ll L(\theta^{m\eta-h})^{-1+k/s}(1+L(\theta^{m\eta-h}u/P)^k). \end{aligned}$$

Then it follows from (3.27) that whenever $\alpha \in \mathfrak{M}$, one has

$$\begin{aligned} &\widehat{f}_{s,\eta}(\alpha;P) - f^*(\alpha) \\ &\quad \ll \sum_{m=M_s+1}^{M_+} \sum_{h=0}^{H_\eta(m)} \sum_{1 \leqslant u \leqslant \theta^{2m+h-m\eta}} L(\theta^{m\eta-h}u)^{-1+k/s}(1+L(\theta^{2m}/P)^k) \\ &\quad \ll L^2 \sum_{m=M_s+1}^{M_+} (\theta^{2m})^{k/s} \sum_{h=0}^{H_\eta(m)} (\theta^{m\eta-h})^{-1}. \end{aligned}$$

On recalling the definitions of M_+, M_s and $H_\eta(m)$, we therefore conclude that

$$\begin{aligned} \widehat{f}_{s,\eta}(\alpha;P) - f^*(\alpha) &\ll L^2 \sum_{m=M_s+1}^{M_+} (\theta^{2m})^{k/s}\theta^{-3m\eta/4} \\ &\ll L^2(P_+)^{k/s-3\eta/8} \ll P^{k/s}L^{-4}. \end{aligned}$$

This establishes the desired approximation to $\widehat{f}_{s,\eta}(\alpha;P)$. The same method, mutatis mutandis, yields the desired conclusion also for $h_{s,\eta}(\alpha;P)$.

Now define

$$Y^*_{s,\eta}(n) = \int_0^1 h^*(\alpha)f^*(\alpha)^{s-1}e(-n\alpha)d\alpha.$$

Lemma 3.7. *One has*

$$Y^+_{s,\eta}(n) - Y^*_{s,\eta}(n) \ll L^{-2^{-3k}/s}.$$

Proof. Trivial estimates for $\widehat{f}_{s,\eta}(\alpha;P)$ and $h_{s,\eta}(\alpha;P)$ yield

$$\widehat{f}_{s,\eta}(\alpha;P) \ll P^{k/s} \quad \text{and} \quad h_{s,\eta}(\alpha;P) \ll P^{k/s}.$$

Thus we find from Lemma 3.6 that, uniformly for $\alpha \in \mathfrak{M}$, one has

$$\begin{aligned} h_{s,\eta}(\alpha;P)\widehat{f}_{s,\eta}(\alpha;P)^{s-1} - h^*(\alpha)f^*(\alpha)^{s-1} &\ll (P^{k/s}L^{-4})(P^{k/s})^{s-1} \\ &\ll P^kL^{-4}. \end{aligned}$$

But the measure of the set of arcs $\mathfrak{M}$ is plainly $O(L^3P^{-k})$, and thus we deduce that

$$\widehat{Y}_{s,\eta}(n;\mathfrak{M}) - Y^*_{s,\eta}(n) \ll \int_{\mathfrak{M}} P^k L^{-4} d\alpha \ll L^{-1}.$$

Finally, from Lemmata 3.2, 3.4 and equation (3.6), we see that

$$Y^+_{s,\eta}(n) - Y^*_{s,\eta}(n) \ll n^{-1/s^3} + L^{-2^{-3k}/s} + L^{-1},$$

and the conclusion of the lemma follows immediately.

We now enter the final stages of the major arc analysis. Write

$$A(q,n) = \sum_{\substack{a=1\\(a,q)=1}}^{q} (q^{-1}S(q,a))^s e(-na/q),$$

and define

$$\mathfrak{S}(n;W) = \sum_{1\leqslant q\leqslant W} A(q,n) \quad \text{and} \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(q,n). \tag{3.28}$$

Also, put

$$J(n;W) = \int_{-W}^{W} \widetilde{v}_1(\beta)\widetilde{v}(\beta)^{s-1} e(-\beta n) d\beta \tag{3.29}$$

and

$$J(n) = \int_{-\infty}^{\infty} \widetilde{v}_1(\beta)\widetilde{v}(\beta)^{s-1} e(-\beta n) d\beta. \tag{3.30}$$

Then on recalling the definitions of $f^*(\alpha)$, $h^*(\alpha)$ and $\mathfrak{M}$, we find that

$$Y^*_{s,\eta}(n) = \mathfrak{S}(n;L)J(n;LP^{-k}). \tag{3.31}$$

The truncated singular series $\mathfrak{S}(n;W)$ defined in (3.28) is the familiar one stemming from the classical analysis of Waring's problem for s kth powers. Since $s \geqslant 4k$ for $k > 3$, one finds by the familiar classical treatment (see Chapters 2 and 4 of Vaughan [10], for example) that

$$\mathfrak{S}(n) - \mathfrak{S}(n;L) \ll L^{-1/k},$$

and further that $1 \ll \mathfrak{S}(n) \ll 1$ uniformly in n. We therefore see from (3.31) that

$$J(n;LP^{-k}) \ll Y^*_{s,\eta}(n) \ll J(n;LP^{-k}). \tag{3.32}$$

In order to extend the truncated singular integral appearing in (3.32) to the full singular integral $J(n)$, we require upper bounds for $\widetilde{v}(\beta)$ and $\widetilde{v}_1(\beta)$.

Lemma 3.8. *For every real number β, one has*

$$\widetilde{v}(\beta) \ll P^{k/s}(1+P^k|\beta|)^{-1/s} \quad \text{and} \quad \widetilde{v}_1(\beta) \ll P^{k/s}(1+P^k|\beta|)^{-1}.$$

Proof. The desired conclusions follow via the argument leading to equation (8.18) of Brüdern and Wooley [1]. We present a reasonably detailed account here, for the sake of completeness. We begin by noting that Theorem 3 on page 400 of Tenenbaum [8] establishes the bound

$$\operatorname{card}(\mathcal{A}^*(Q,R)) \ll Q(\log R)^{-1}, \tag{3.33}$$

uniformly for $Q \geqslant \sqrt{R} \geqslant 2$. But for $0 < Q < \sqrt{R}$, the set $\mathcal{A}^*(Q,R)$ contains at most the element 1, and thus the aforementioned bound remains valid for all positive numbers Q. Observe next that whenever $\phi \in \mathbb{R} \setminus \{0\}$, it follows via partial integration that whenever $0 < \delta < 1$,

$$\begin{aligned} v_\delta(\phi;W) &= (2\pi ik\phi)^{-1}\left(W^{1-k-\delta}e(\phi W^k) - (\theta^{-1}W)^{1-k-\delta}e(\phi(\theta^{-1}W)^k)\right) \\ &\quad - (2\pi ik\phi)^{-1}\int_{\theta^{-1}W}^{W}(1-k-\delta)\gamma^{-k-\delta}e(\phi\gamma^k)d\gamma \\ &\ll W^{1-\delta}(W^k|\phi|)^{-1}. \end{aligned}$$

On combining this estimate with a trivial one, therefore, we obtain the upper bound

$$v_{1-k/s}(\phi;W) \ll W^{k/s}(1+W^k|\phi|)^{-1}, \tag{3.34}$$

uniformly for $\phi \in \mathbb{R}$.

Next, for $\phi > 0$ and $Q \geqslant T \geqslant 1$, define

$$\mathcal{U}(\phi;Q,T,R) = \sum_{u \in \mathcal{B}(Q/T,R)} u^{-1+k/s}T^{k/s}(1+T^k u^k \phi)^{-1}.$$

Whenever $u \in \mathcal{B}(Q/T,R)$, one has $u > \theta^{-1}Q/T$, and thus we see that

$$\mathcal{U}(\phi;Q,T,R) \ll (Q/T)^{-1+k/s}T^{k/s}(1+Q^k\phi)^{-1}\operatorname{card}\,(\mathcal{A}^*(Q/T,R)),$$

whence by (3.33),

$$\mathcal{U}(\phi;Q,T,R) \ll Q^{k/s}(\log R)^{-1}(1+Q^k\phi)^{-1}. \tag{3.35}$$

On substituting the upper bound (3.35) into (3.25), we obtain from the estimate (3.34) the new bound

$$\begin{aligned} \widetilde{v}(\beta) &\ll \sum_{m=M_s+1}^{M_+} \sum_{h=0}^{H_\eta(m)} \mathcal{U}(\beta;\theta^{2m},\theta^{m\eta-h},\theta^{2m\eta}) \\ &\ll \sum_{m=M_s+1}^{M_+} \sum_{h=0}^{H_\eta(m)} (\theta^{2m})^{k/s}m^{-1}(1+\theta^{2mk}|\beta|)^{-1}. \end{aligned}$$

But $H_\eta(m) = O(m)$, and so it follows that

$$\widetilde{v}(\beta) \ll \sum_{m=M_s+1}^{M_+} \theta^{2mk/s}(1+\theta^{2mk}|\beta|)^{-1}. \tag{3.36}$$

When $|\beta| \leqslant \theta^{-2M_+k}$, we find from the latter inequality that

$$\widetilde{v}(\beta) \ll \sum_{m=M_s+1}^{M_+} \theta^{2mk/s} \ll P_+^{k/s} \ll P^{k/s}.$$

When $|\beta| > \theta^{-2M_+k}$, meanwhile, we define M_0 via the relation $|\beta| = \theta^{-2M_0k}$, and we deduce that

$$\begin{aligned}\widetilde{v}(\beta) &\ll \sum_{M_s+1\leqslant m\leqslant M_0} \theta^{2mk/s} + \sum_{M_0<m\leqslant M_+} \theta^{2mk/s-2mk}|\beta|^{-1}\\ &\ll \theta^{2M_0k/s} + (\theta^{2M_0k})^{-1+1/s}|\beta|^{-1} \ll |\beta|^{-1/s}.\end{aligned}$$

Then in any case, one has

$$\widetilde{v}(\beta) \ll P^{k/s}(1+P^k|\beta|)^{-1/s},$$

and this yields the first estimate of the lemma.

In order to estimate $\widetilde{v}_1(\beta)$, we follow the same argument as that above, but now we obtain the estimate

$$\widetilde{v}_1(\beta) \ll \sum_{m=M_-+1}^{M_+} \theta^{2mk/s}(1+\theta^{2mk}|\beta|)^{-1}$$

in place of (3.36). However, since $M_+ - M_- \ll 1$, we immediately deduce that

$$\widetilde{v}_1(\beta) \ll P^{k/s}(1+P^k|\beta|)^{-1},$$

and so the proof of the lemma is complete.

We now return to the analysis of the singular integral $J(n; LP^{-k})$. Observe first that on substituting the conclusion of Lemma 3.8 into (3.29) and (3.30), one finds that

$$J(n) - J(n; LP^{-k}) \ll P^k \int_{LP^{-k}}^{\infty} (1+P^k\beta)^{-2+1/s}d\beta \ll L^{-1+1/s}. \tag{3.37}$$

Similarly, one has

$$J(n) \ll P^k \int_0^{\infty} (1+P^k\beta)^{-2+1/s}d\beta \ll 1, \tag{3.38}$$

so that the singular integral is absolutely convergent. On substituting (3.37) and (3.38) into (3.32), we conclude thus far that

$$Y^*_{s,\eta}(n) \ll 1 + O(L^{-1+1/s}) \ll 1. \tag{3.39}$$

In order to bound from below the singular integral $J(n)$, we require an estimate for the cardinality of the set $\mathcal{A}^*(Q,R)$. Here we apply work of Friedlander [5]. Suppose that A, B and C are fixed real numbers with $B > A \geqslant 1$ and $C > 0$. Let Q and R be large real numbers satisfying $R^A \leqslant Q \leqslant R^B$. Then as an immediate consequence of Theorem 1 of Friedlander [5], one has the bounds

$$\frac{CQ}{\log R} \ll_{A,B} \mathrm{card}(\mathcal{A}^*((1+C)Q,R)) - \mathrm{card}(\mathcal{A}^*(Q,R)) \ll_{A,B} \frac{CQ}{\log R}. \tag{3.40}$$

Observe next that on making a change of variable in (3.22), it follows that for each positive number ζ, one has

$$v_{1-k/s}(\zeta^k\beta;W) = \zeta^{-k/s}\int_{\theta^{-1}\zeta W}^{\zeta W} \gamma^{-1+k/s} e(\beta\gamma^k)d\gamma. \tag{3.41}$$

Define $J^*(n) = J^*(n;\mathbf{Z})$ by

$$J^*(n;\mathbf{Z}) = \int_{-\infty}^{\infty}\int_{\mathfrak{B}(\mathbf{Z})} (\gamma_1\dots\gamma_s)^{-1+k/s} e(\beta(\gamma_1^k+\dots+\gamma_s^k-n))d\boldsymbol{\gamma}d\beta, \tag{3.42}$$

where

$$\mathfrak{B}(\mathbf{Z}) = [\theta^{-1}Z_1, Z_1]\times[\theta^{-1}Z_2,Z_2]\times\dots\times[\theta^{-1}Z_s,Z_s].$$

Then with $\mathbf{Z} = \mathbf{Z}(\mathbf{m},\mathbf{h},\mathbf{u})$ defined by

$$Z_i = u_i\theta^{m_i\eta-h_i} \quad (1\leqslant i\leqslant s),$$

it follows from (3.30) together with (3.25), (3.26) and (3.41) that

$$J(n) = \sum_{\mathbf{m},\mathbf{h},\mathbf{u}} (u_1\dots u_s)^{-1}J^*(n;\mathbf{Z}), \tag{3.43}$$

where the summation is over

$$M_- + 1 \leqslant m_1 \leqslant M_+, \quad M_s+1\leqslant m_i\leqslant M_+ \quad (2\leqslant i\leqslant s), \tag{3.44}$$

$$0\leqslant h_j\leqslant H_\eta(m_j) \quad \text{and} \quad u_j\in\mathcal{B}(\theta^{2m_j+h_j-m_j\eta},\theta^{2m_j\eta}) \quad (1\leqslant j\leqslant s). \tag{3.45}$$

Consider next the set Σ of ordered pairs $(\sigma,\tau)\in\mathbb{R}^2$ with

$$s^{-1/k} < \sigma < 1, \quad 0<\tau<1 \quad \text{and} \quad \sigma^k + (s-1)\tau^k = 1. \tag{3.46}$$

We aim to show that for some $(\sigma,\tau)\in\Sigma$, there exists an s-tuple $\mathbf{m}$ satisfying (3.44) for which one has

$$\theta^{2m_1-7/5}<\sigma P<\theta^{2m_1-3/5}\quad\text{and}\quad \theta^{2m_j-7/5}<\tau P<\theta^{2m_j-3/5}\quad (2\leqslant j\leqslant s). \tag{3.47}$$

Note first that there is a choice of σ_0 with $\theta^{-2}6^{-1/k}\leqslant \sigma_0<6^{-1/k}$ for which the first condition of (3.47) is met with $\sigma=\sigma_0$. Define the positive number τ_0 to be the corresponding solution τ of the equation in (3.46). Then plainly $0<\tau_0<1$. It is possible that the second condition of (3.47) is already met with $\tau=\tau_0$, in which case we are done. Otherwise, we consider the sequence of ordered pairs $(\sigma_i,\tau_i)\in\mathbb{R}^2$ defined by

$$\sigma_{i+1}=\theta^{-2}\sigma_i,\quad \tau_{i+1}=(1-\sigma_{i+1}^k)^{1/k}(s-1)^{-1/k}\quad (0\leqslant i<5).$$

It is simple to verify that each pair $(\sigma,\tau)=(\sigma_i,\tau_i)$ $(0\leqslant i<5)$, defined in this way, satisfies the conditions (3.46). Moreover, by construction, the first condition of (3.47) is met with $\sigma=\sigma_i$ for each i. Also, since $|\theta^{-2k}-1|<10^{-100}$ and

$$1/5-10^{-10}<\frac{\sigma_i^k}{1-\sigma_i^k}<1/5,$$

it is apparent that for each i with $0\leqslant i<5$, one has

$$\frac{\tau_{i+1}}{\tau_i}=\left(\frac{1-(\theta^{-2}\sigma_i)^k}{1-\sigma_i^k}\right)^{1/k}<(1+0.401\times 10^{-100}k^{-1})^{1/k}<\theta^{0.41},$$

and similarly

$$\tau_{i+1}/\tau_i>(1+0.399\times 10^{-100}k^{-1})^{1/k}>\theta^{0.39}.$$

It follows that for some i with $0\leqslant i<5$, there is an integer m for which

$$\theta^{2m-5/4}<\tau_iP<\theta^{2m-3/4},$$

whence the second condition of (3.47) is satisfied with $\tau=\tau_i$. This establishes the existence of the desired pair $(\sigma,\tau)\in\Sigma$.

With the fixed choice of $\mathbf{m}$ provided by the above discussion, and an arbitrary choice of $\mathbf{h}$ satisfying (3.45), we find that whenever

$$\theta^{2m_i-3/5+h_i-m_i\eta}<u_i<\theta^{2m_i-2/5+h_i-m_i\eta}\quad (1\leqslant i\leqslant s), \tag{3.48}$$

then

$$[\theta^{2m_1-7/5},\theta^{2m_1-3/5}]\times[\theta^{2m_2-7/5},\theta^{2m_2-3/5}]\times\cdots\times[\theta^{2m_s-7/5},\theta^{2m_s-3/5}]\subseteq\mathfrak{B}(\mathbf{Z}).$$

An application of Fourier's integral formula to (3.42) establishes in such circumstances that

$$\begin{aligned}J^*(n;\mathbf{Z})&\gg (Z_1\dots Z_s)^{-1+k/s}(Z_1\dots Z_s/n)\\ &\gg (P^s)^{-1+k/s}P^{s-k}=1.\end{aligned}$$

Then we may conclude from (3.43) that

$$J(n) \gg \sum_{\mathbf{h},\mathbf{u}} (u_1 \dots u_s)^{-1},$$

where the summation is over values of $\mathbf{h}$ and $\mathbf{u}$ satisfying (3.45) and (3.48), for our fixed selection of $\mathbf{m}$. We therefore find from (3.40) that

$$\begin{aligned} J(n) &\gg \prod_{i=1}^{s} \left(\sum_{h=0}^{H_\eta(m_i)} (\theta^{-2m_i-h_i+m_i\eta})(\theta^{2m_i+h_i-m_i\eta}/m_i) \right) \\ &\gg \prod_{i=1}^{s} (m_i^{-1} H_\eta(m_i)). \end{aligned}$$

Finally, on recalling that $H_\eta(m_i) \gg m_i$, we conclude that $J(n) \gg 1$, whence by (3.37), we also have $J(n; LP^{-k}) \gg 1$. The relation (3.32) consequently yields the desired conclusion

$$Y^*_{s,\eta}(n) \gg 1. \tag{3.49}$$

On recalling (3.1) and the conclusion of Lemma 3.7, it is apparent from the estimates (3.39) and (3.49) that

$$1 \ll Y_{s,\eta}(n) \ll 1,$$

and so the conclusion of Theorem 3.1 has finally been established.

4. Thin bases in random sets. We are now in a position from which we may apply the ideas of Vu [14] so as to establish Theorem 1.1. There are sufficiently many differences between the framework developed in §§2 and 3 here, and that found in Vu [14], that a reasonably detailed account seems appropriate. However, we will economise where possible.

We fix natural numbers $k \geqslant 3$ and $s \geqslant \mathfrak{G}(k)$, and we take η to be a positive number, sufficiently small in terms of s and k (in the context of §3). We define a random subset $\mathfrak{X} = \mathfrak{X}_{k,s}(\eta)$ of $\mathbb{N}_0^k$ by including in $\mathfrak{X}$, for each large integer $x \in \mathcal{A}_\eta$, the number x^k with probability

$$p_x = cx^{-1+k/s}(\log x)^{1/s}. \tag{4.1}$$

Here, the number c is a positive constant to be fixed later. We take t_x to be the characteristic random variable indicating the choice of x^k, so that $t_x = 1$ when x^k is included in $\mathfrak{X}$, and $t_x = 0$ otherwise. In particular, one has $\Pr(t_x = 1) = p_x$ and $\Pr(t_x = 0) = 1 - p_x$, and furthermore the t_x are independent for $x \in \mathcal{A}_\eta$. Following Vu [14], we express the number of representations of n as the sum of s kth powers of elements of $\mathfrak{X}$ as the random variable

$$R^s_{\mathfrak{X}}(n) = \sum_{\mathbf{x}} \prod_{j=1}^{s} t_{x_j}, \tag{4.2}$$

where the summation is over $x_j \in \mathcal{A}_\eta$ $(1 \leqslant j \leqslant s)$, with

$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_s, \tag{4.3}$$

and satisfying the equation

$$x_1^k + \cdots + x_s^k = n. \tag{4.4}$$

We write $\mathcal{G}(\mathbf{t})$ for the polynomial on the right hand side of (4.2), and we note that this polynomial depends only on the variables t_y with $y \in \mathcal{A}_\eta \cap [1, n^{1/k}]$.

In order to make use of Vu's concentration lemma so as to establish Theorem 1.1, we must introduce some further notation. Let $t_1, \ldots, t_m$ be independent $\{0,1\}$-random variables, and consider a polynomial $F(\mathbf{t}) = F(t_1, \ldots, t_m)$ of degree d. We say that F is *positive* if all of its non-zero coefficients are positive, and we say that F is *normal* if its coefficients are at most 1 in size. For each set A of at most d indices, with possible repetitions, we write $\partial_A(F)$ for the partial derivative of F with respect to the variables given by the indices in A. We also write $\mathbb{E}(F)$ for the expectation of F, and $\mathbb{E}_A(F)$ for $\mathbb{E}(\partial_A F)$.

Lemma 4.1. *When s is a fixed natural number, and k, α and ε are fixed positive numbers, there is a positive number $g = g(s, k, \alpha, \varepsilon)$ satisfying the following property. Whenever $F(t_1, \ldots, t_n)$ is a positive, normal, homogeneous polynomial of degree s satisfying*

(1) $\mathbb{E}(F) > g \log n$,

(2) *for all sets of indices A with $1 \leqslant card(A) \leqslant s-1$, one has $\mathbb{E}_A(F) < n^{-\alpha}$,*

then one has

$$\Pr(|F - \mathbb{E}(F)| > \varepsilon \mathbb{E}(F)) < n^{-2k}.$$

Proof. This is Vu's concentration lemma (see Lemma 1.3 of Vu [14]).

Finally, we recall the Borel-Cantelli Lemma (this is Lemma 1.5 of Vu [14]).

Lemma 4.2. *Let $(A_i)_{i=1}^\infty$ be a sequence of events in a probability space. Suppose that the series $\sum_{i=1}^\infty Pr(A_i)$ converges. Then with probability 1, at most a finite number of the events A_i can occur.*

We now initiate our main assault on the proof of Theorem 1.1. We divide the solutions underlying $R_{\mathfrak{X}}^s(n)$ into two types, namely those that are "small" and those that are "typical". Recall the definition of P_s from §3. We let $\mathcal{S}^+(n)$ denote the set of s-tuples $(x_1, \ldots, x_s) \in \mathcal{A}_\eta^s$ satisfying (4.3), (4.4) and $x_1 > P_s$, and let $\mathcal{S}^0(n)$ denote the corresponding set of s-tuples with the last inequality replaced by $x_1 \leqslant P_s$. Also, let $\mathcal{S}_{\mathfrak{X}}^+(n)$ denote the set corresponding to $\mathcal{S}^+(n)$ wherein we restrict the s-tuples $\mathbf{x}$ to the set $\mathfrak{X}^s$, and likewise for $\mathcal{S}_{\mathfrak{X}}^0(n)$. Finally, define

$$R_{\mathfrak{X},s}^+(n) = \sum_{\mathbf{x} \in \mathcal{S}_{\mathfrak{X}}^+(n)} \prod_{j=1}^s t_{x_j} \quad \text{and} \quad R_{\mathfrak{X},s}^0(n) = \sum_{\mathbf{x} \in \mathcal{S}_{\mathfrak{X}}^0(n)} \prod_{j=1}^s t_{x_j},$$

so that by (4.2) one has

$$R^s_{\mathfrak{X}}(n) = R^+_{\mathfrak{X},s}(n) + R^0_{\mathfrak{X},s}(n). \tag{4.5}$$

It is convenient to write $F^+(\mathbf{t})$ for the polynomial $R^+_{\mathfrak{X},s}(n)$, and similarly $F^0(\mathbf{t})$ for $R^0_{\mathfrak{X},s}(n)$.

In preparation for an application of Lemma 4.1, we estimate the expectations of $F^+(\mathbf{t})$ and its partial derivatives. In what follows, we suppose that n is sufficiently large in terms of s, k and η.

Lemma 4.3. *One has $\mathbb{E}(F^+) \asymp c^s \log n$. Here, the constants implicit in Vinogradov's notation depend at most on s, k and η.*

Proof. On recalling (4.1), we see that

$$\mathbb{E}(F^+(\mathbf{t})) = c^s \sum_{x_1,\dots,x_s\in\mathcal{A}_\eta} \prod_{i=1}^{s} x_i^{-1+k/s}(\log x_i)^{1/s},$$

where the summation is subject to (4.3), (4.4) and $x_1 > P_s$. The last condition, together with (4.3), ensures that $\log x_i \asymp \log n$ for $1 \leqslant i \leqslant s$, so that on considering the diophantine equation underlying (3.4), we see that

$$\mathbb{E}(F^+) \asymp c^s \widehat{Y}_{s,\eta}(n) \log n. \tag{4.6}$$

It may be worth remarking here that in order to account for the condition (4.3), one may need to reorder variables. Such permutations, however, lead to implicit factors of at most $s!$ within (4.6). But by Theorem 3.1 together with Lemma 3.2, there exist positive numbers $\Xi_\pm$, independent of n and c, with the property that

$$\Xi_- + O(n^{-1/s^3}) \leqslant \widehat{Y}_{s,\eta}(n) \leqslant \Xi_+ + O(n^{-1/s^3}).$$

The conclusion of the lemma is therefore immediate from (4.6).

Lemma 4.4. *Let A be a set of indices $\{i_1,\dots,i_r\}$ with $i_j \in \mathcal{A}_\eta \cap \{1,2,\dots,[n^{1/k}]\}$ $(1 \leqslant j \leqslant r)$, and with $1 \leqslant r \leqslant s-1$. Then one has $\mathbb{E}_A(F^+) \ll c^s n^{-1/s^3}$.*

Proof. With A defined as in the statement of the lemma, we write

$$m = n - \sum_{y\in A} y^k$$

and $l = s - \mathrm{card}(A)$. From the definition of F^+, one has

$$\partial_A F^+ = \sum_{\substack{\mathbf{x}\in\mathcal{S}^+_{\mathfrak{X}}(n)\\ A\subset\mathbf{x}}} \prod_{x_j\in\mathbf{x}\setminus A} t_{x_j},$$

whence from (4.1),

$$\mathbb{E}_A(F^+) \ll c^l \sum_{x_1,\ldots,x_l \in \mathcal{A}_\eta} \prod_{i=1}^{l} x_i^{-1+k/s} (\log x_i)^{1/s},$$

where the summation is subject to

$$x_l \geqslant x_{l-1} \geqslant \ldots \geqslant x_1 > P_s \tag{4.7}$$

and

$$x_1^k + \cdots + x_l^k = m. \tag{4.8}$$

It follows from (4.8) that $x_i \leqslant m^{1/k}$ $(1 \leqslant i \leqslant l)$ and $x_l \geqslant (m/l)^{1/k}$, and from (4.7) we see that $m > P_s^k \gg n^{1/s}$. Then on noting that the above inequality now yields

$$\mathbb{E}_A(F^+) \ll c^l \log m \sum_{x_1,\ldots,x_l \in \mathcal{A}_\eta} (x_1 \ldots x_l)^{-1+k/s},$$

under the same conditions, it follows by considering the underlying diophantine equations that

$$\mathbb{E}_A(F^+) \ll c^l \log m \int_0^1 h_{s,\eta}(\alpha; m^{1/k}) f_{s,\eta}(\alpha; m^{1/k})^{l-1} e(-\alpha m) d\alpha.$$

An application of Hölder's inequality, followed by use of Lemmata 2.1 and 2.2, leads to the upper bound

$$\begin{aligned}
\mathbb{E}_A(F^+) &\ll (c^l \log m) \Big(\int_0^1 |h_{s,\eta}(\alpha; m^{1/k})|^{s-1} d\alpha \Big)^{1/(s-1)} \\
&\quad \times \Big(\int_0^1 |f_{s,\eta}(\alpha; m^{1/k})|^{s-1} d\alpha \Big)^{(l-1)/(s-1)} \\
&\ll (c^l \log m)(m^{-1/s})^{1/(s-1)} ((\log m)^{s-1})^{(l-1)/(s-1)} \\
&\ll c^l m^{-1/s^2}.
\end{aligned}$$

Consequently, our earlier observation that $m \gg n^{1/s}$ delivers the conclusion

$$\mathbb{E}_A(F^+) \ll c^l n^{-1/s^3}.$$

This completes the proof of the lemma.

We are now equipped to apply Lemma 4.1. We take s and k as above, set $\alpha = 1/s^4$, and choose ε to be a sufficiently small positive number. Let $g = g(s,k,\alpha,\varepsilon)$ be the number implicitly defined in the statement of Lemma 4.1. Then if we choose c large enough, we find that Lemma 4.3 guarantees that $\mathbb{E}(F^+) > g \log n$, and

Lemma 4.4 shows that for all sets of indices A with $1 \leqslant \mathrm{card}(A) \leqslant s-1$, one has $\mathbb{E}_A(F^+) < n^{-\alpha}$. Then Lemma 4.1 demonstrates that

$$\Pr(|F^+ - \mathbb{E}(F^+)| > \varepsilon \mathbb{E}(F^+)) < n^{-2}.$$

But $\sum_{n=1}^{\infty} n^{-2}$ converges, and so Lemma 4.2 implies that with probability 1, the chain of inequalities

$$(1-\varepsilon)\mathbb{E}(F^+) \leqslant F^+ \leqslant (1+\varepsilon)\mathbb{E}(F^+)$$

fails to hold on at most finitely many occasions. In particular, with probability 1, one has

$$R^+_{\mathfrak{X},s}(n) \asymp \log n. \tag{4.9}$$

It remains to consider the contribution of $R^0_{\mathfrak{X},s}(n)$ within (4.5).

Lemma 4.5. *One has* $\mathbb{E}(F^0) \ll c^s n^{-1/s^4}$.

Proof. On recalling the definition of $F^0(\mathbf{t})$, it follows from (4.1) that

$$\mathbb{E}(F^0) \ll c^s \sum_{x_1,\dots,x_s \in \mathcal{A}_\eta} \prod_{i=1}^{s} x_i^{-1+k/s} (\log x_i)^{1/s},$$

where the summation is subject to (4.3), (4.4) and $x_1 < P_s$. But it follows from (4.4) that $x_i \leqslant n^{1/k}$ $(1 \leqslant i \leqslant s)$ and $x_s \geqslant (n/s)^{1/k}$. Then on noting that the above inequality yields

$$\mathbb{E}(F^0) \ll c^s \log n \sum_{x_1,\dots,x_s \in \mathcal{A}_\eta} (x_1 \dots x_s)^{-1+k/s},$$

under the same summation conditions, it follows by considering the underlying diophantine equations that

$$\mathbb{E}(F^0) \ll c^s \log n \int_0^1 h_{s,\eta}(\alpha;P) f_{s,\eta}(\alpha;P)^{s-2} f_{s,\eta}(\alpha;P^{1/s}) e(-n\alpha)\, d\alpha.$$

Then it follows from Lemma 2.3 that

$$\begin{aligned}\mathbb{E}(F^0) &\ll c^s \log n \int_0^1 |h_{s,\eta}(\alpha;P) f_{s,\eta}(\alpha;P)^{s-2} f_{s,\eta}(\alpha;P^{1/s})|\, d\alpha \\ &\ll (c^s \log n) n^{-1/s^3} \ll c^s n^{-1/s^4}.\end{aligned}$$

This completes the proof of the lemma.

The proof of Theorem 1.1. Equipped with the analogue of Lemma 3.6 of Vu [14] provided by Lemma 4.5, the argument bridging pages 128 and 129 of Vu [14] may be applied to show that there is a positive constant C such that, with probability at least 4/5, one has $R^0_{\mathfrak{X},s}(n) \leqslant C$ for every natural number n. Combined with our earlier conclusion (4.9), we see that there exists a sequence $\mathfrak{X}$ and a finite number N_0 such that $R^s_{\mathfrak{X}}(n) \asymp \log n$ for all $n \geqslant N_0$. In order to complete the proof of Theorem 1.1, we note merely that since

$$\sum_{n\leqslant st} R^s_{\mathfrak{X}}(n) \ll t\log t$$

and

$$\sum_{n\leqslant st} R^s_{\mathfrak{X}}(n) \gg (\mathrm{card}(\mathfrak{X}\cap[1,t]))^s,$$

then for all large numbers t one has

$$\mathrm{card}(\mathfrak{X}\cap[1,t]) \ll (t\log t)^{1/s}.$$

Corollary 1 to Theorem 1.1 follows from (1.6) via a modest computation, and Corollary 2 follows from Theorem 1.1 via an immediate argument of Vu [14]. As a simple variant of the latter, we proceed as follows. Suppose that $s \geqslant g(k)$, whence, in particular, one has $s \geqslant \mathfrak{G}(k)$. Our proof of Theorem 1.1 shows that there is a thin set $\mathfrak{X}$ for which there is a natural number N_0, satisfying the property that whenever $n \geqslant N_0$, then one has $R^s_{\mathfrak{X}}(n) \asymp \log n$. We now modify our definition of $\mathcal{A}_\eta$, replacing it with the set

$$\mathcal{A}_\eta \cup \{0,1,2,\dots,[N_0^{1/k}]\},$$

and we revise the definition of the probability function p_x by setting $p_x = 1$ for $0 \leqslant x \leqslant [N_0^{1/k}]$. The latter has the effect of ensuring that the set $\mathfrak{X}$ automatically contains the elements $0, 1^k, \dots, [N_0^{1/k}]^k$. Since this adjustment affects only finitely many (small) elements of $\mathfrak{X}$, one finds that the discussion of §§2 and 3, as well as this section, remains valid. In this way we establish the existence of a set $\widehat{\mathfrak{X}}$, with

$$\mathrm{card}(\widehat{\mathfrak{X}}\cap[1,t]) \ll (t\log t)^{1/s}$$

for each positive number t, and satisfying the condition that $R^s_{\widehat{\mathfrak{X}}}(n) \asymp \log n$ for all $n \geqslant N_0$. Moreover, it follows from the definition of $g(k)$ that this set $\widehat{\mathfrak{X}}$ also satisfies the property that whenever $1 \leqslant n \leqslant N_0$, one has

$$R^s_{\widehat{\mathfrak{X}}}(n) \geqslant 1 \gg \log n.$$

Then $\widehat{\mathfrak{X}}$ is a basis of order s, and this completes the proof of Corollary 2 to Theorem 1.1.

5. Thin bases for smaller exponents, and related problems. The treatment of §§2–4 above was somewhat general, in order that some measure of concision be achieved. However, an expert in the modern smooth number variants of the circle method will rapidly discern a simple underlying pattern. Suppose that, whenever η is a positive number sufficiently small in terms of k, one has the bound

$$\int_0^1 \Big| \sum_{x\in\mathcal{A}(P,P^\eta)} e(\alpha x^k)\Big|^{2u} d\alpha \ll P^{2u-k}. \tag{5.1}$$

Then one may establish the conclusion of Theorem 1.1 with $\mathfrak{G}(k)$ replaced by $2u+1$. In particular, combining the available mean values of the shape (5.1) from the work of Vaughan [9] and Vaughan and Wooley [12], [13], one may establish the following theorem.

Theorem 5.1. *Let $\mathfrak{H}(k)$ denote the number appearing in the table below. Then whenever $3\leqslant k\leqslant 20$ and $s\geqslant \mathfrak{H}(k)$, there exists a subset $\mathfrak{X}_k=\mathfrak{X}_k(s)$ of $\mathbb{N}_0^k$, with cardinality satisfying the condition* (1.2)*, and such that, when n is sufficiently large in terms of k and s, one has*

$$\log n \ll R_s(n;\mathfrak{X}_k) \ll \log n.$$

k	3	4	5	6	7	8	9	10	11	12
$\mathfrak{H}(k)$	9	13	19	27	35	43	51	61	69	77

k	13	14	15	16	17	18	19	20
$\mathfrak{H}(k)$	85	93	101	111	119	127	135	143

Here, when $k=4$, the tabulated value $\mathfrak{H}(4)=13$ indicates that whenever $s\geqslant 13$, the above conclusion holds whenever n satisfies the additional condition that $n\equiv r$ (mod 16) *for some integer r with $1\leqslant r\leqslant s$.*

We note that the tabulated entry $\mathfrak{H}(6)=27$ can be reduced to $\mathfrak{H}(6)=25$ with some technical effort. The problem here is that one has only the estimate

$$\int_0^1 \Big| \sum_{x\in\mathcal{A}(P,P^\eta)} e(\alpha x^6)\Big|^{24} d\alpha \ll P^{18+\varepsilon},$$

valid for each $\varepsilon>0$, available from work of Vaughan and Wooley [12]. However, technology described in Vaughan and Wooley [11] can be adapted in the present context to yield a serviceable substitute for (5.1) with $k=6$ and $u=24$.

It may also be useful to point out that, for those enthusiasts of an exotic disposition, the methods herein are easily adapted to restricted sets. All that is necessary to make progress is a set $\mathcal{H}\subseteq\mathbb{N}$, well-distributed in arithmetic progressions modulo q to a height at least as large as $(\log x)^A$ for elements of size x, with A sufficiently large, and for which one has a mean value estimate of the shape

$$\int_0^1 \Big| \sum_{x\in\mathcal{H}\cap[1,Q]} e(\alpha x^k)\Big|^{2u} d\alpha \ll Q^{-k}(\mathrm{card}(\mathcal{H}\cap[1,Q]))^{2u}.$$

There should be no difficulty, for example, in proving that for $k \geqslant 1$, there exist thin sets of kth powers of prime numbers that provide asymptotic bases. Similar comments hold, mutatis mutandis, when kth powers are replaced by more general polynomials.

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