ON DIOPHANTINE INEQUALITIES: FREEMAN'S ASYMPTOTIC FORMULAE

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1. Introduction. It is only within the past couple of years that the Davenport-Heilbronn method, now in its second half-century of life, has delivered asymptotic formulae for the number of solutions of diophantine inequalities in many variables. Let k and s be positive integers with $k \geqslant 2$ and $s \geqslant 2^k+1$, and let τ be any positive number. Then whenever $\lambda_1,\dots,\lambda_s$ are non-zero real numbers, not all in rational ratio, and not all of the same sign in the case that k is even, it follows from Davenport and Heilbronn's seminal paper [12] that there exist arbitrarily large non-zero integral solutions $\mathbf{x}$ of the diophantine inequality

$$|\lambda_1x_1^k+\dots+\lambda_sx_s^k|<\tau. \tag{1.1}$$

If we write $N(P)$ to denote the number of solutions of the inequality (1.1) with $\mathbf{x}\in[-P,P]^s\cap\mathbb{Z}^s$, then the method of Davenport and Heilbronn [12] establishes that $N(P)\gg P^{s-k}$ for arbitrarily large values of P. However, an inescapable feature of their method forces the latter values of P to be determined from the convergents to the continued fraction expansion of some suitable ratio λ_i/λ_j $(i\neq j)$, and consequently the sequence of permissible values of P may be arbitrarily sparse. This limitation permeates the subsequent literature on the topic (see, for example, Davenport and Roth [13] and Brüdern and Cook [7]). Inspired by work of Bentkus and Götze [2] on the value distribution of positive definite quadratic forms, Freeman [16] has very recently developed a variant of the Davenport-Heilbronn method in which an asymptotic formula for $N(P)$ is established for *all* values of P large enough in terms of k, s, τ and $\boldsymbol{\lambda}$. Our purpose in this paper is to modestly sharpen the conclusions available from Freeman's variant of the Davenport-Heilbronn method, with the parallel objective of increasing the flexibility of the method so as to bring familiar targets within range of the new technology.

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In order to state our conclusions we require some notation. Here we temporarily indulge in some technical issues in order that our results have wider application. Consider a fixed integer $k \geqslant 2$, and write

$$f(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^k), \tag{1.2}$$

where, as usual, we denote $e^{2\pi i z}$ by $e(z)$. In order to discuss our application of the circle method, we define a typical Hardy-Littlewood dissection as follows. We take the set of major arcs $\mathfrak{N}(Q)$ to be the union of the intervals

$$\mathfrak{N}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \leqslant QP^{-k}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a, q) = 1$, and then denote the corresponding set of minor arcs by $\mathfrak{n}(Q) = [0, 1) \setminus \mathfrak{N}(Q)$. We refer to a positive number $u > 2k$ as being *accessible to the exponent* k when there exist increasing functions $S_i(P)$ $(i = 1, 2)$, with the property that (a) when P is large, one has $2 \leqslant S_i(P) \leqslant P$ $(i = 1, 2)$, (b) the functions $S_i(P)$ increase monotonically to infinity as $P \to \infty$ $(i = 1, 2)$, and (c) whenever $t \geqslant u$, one has

$$\int_{\mathfrak{n}(S_1(P))} |f(\alpha)|^t d\alpha \ll S_2(P)^{-1} P^{t-k}. \tag{1.3}$$

Write $N(P) = N_\tau(P; s, k; \boldsymbol{\lambda})$ for the number of integral solutions of the inequality (1.1) with $\mathbf{x} \in [1, P]^s$. We observe, in passing, that the restriction of the solution set to the positive quadrant represents no serious constraint, since we may take the union of the solution sets corresponding to suitable coefficient s-tuples $(\pm\lambda_1, \ldots, \pm\lambda_s)$ in order to recover the box $[-P, P]^s$ (of course, there may be solutions also in which $x_i = 0$ for some i, but these may be expected to contribute a number of solutions with smaller order of magnitude than the anticipated main term in the sought-after asymptotic formula). We assume throughout that no coefficient λ_i is zero. It is convenient also to put $\mu_i = |\lambda_i|$ and $\sigma_i = \lambda_i/\mu_i$ for $1 \leqslant i \leqslant s$. Finally, we define the singular integral $\Omega = \Omega(s, k; \boldsymbol{\lambda})$ by taking

$$\Omega(s, k; \boldsymbol{\lambda}) = k^{-s} |\lambda_1 \ldots \lambda_s|^{-1/k} C(s, k; \boldsymbol{\lambda}),$$

where

$$C(s, k; \boldsymbol{\lambda}) = \int_{\mathfrak{B}} (v_1 \ldots v_{s-1})^{1/k-1} ((-\sigma_s)(\sigma_1 v_1 + \cdots + \sigma_{s-1} v_{s-1}))^{1/k-1} d\mathbf{v},$$

and $\mathfrak{B}$ denotes the subset of the box $[0, \mu_1] \times \cdots \times [0, \mu_{s-1}]$ satisfying the condition that $-\sigma_s(\sigma_1 v_1 + \cdots + \sigma_{s-1} v_{s-1}) \in [0, \mu_s]$. We note that $\Omega(s, k; \boldsymbol{\lambda}) > 0$ provided only that $\sigma_1, \ldots, \sigma_s$ are not all of the same sign.

Theorem 1.1. *Whenever s is an integer accessible to the exponent k, one has the asymptotic formula*

$$N_\tau(P;s,k;\boldsymbol{\lambda}) = 2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k} + o(P^{s-k}). \tag{1.4}$$

An asymptotic formula of the type (1.4) has been established by Freeman [16] for $s \geqslant 2^k+1$. By combining mean value estimates of Vaughan [23], [24], Boklan [6] and Ford [15], one obtains the following refinement of Freeman's conclusion.

Corollary. *The asymptotic formula* (1.4) *holds whenever $s \geqslant 2^k$ $(k \geqslant 3)$, whenever $s \geqslant \frac{7}{8}2^k$ $(k \geqslant 6)$, and whenever $s \geqslant k^2(\log k + \log\log k + O(1))$ when k is large.*

One might rather crudely summarise the prerequisites for a successful application of Freeman's variant of the Davenport-Heilbronn method as being (i) a "clean" mean value estimate of the asymptotically sharp shape

$$\int_0^1 |f(\alpha)|^t d\alpha \ll P^{t-k}, \tag{1.5}$$

and (ii) the weak analogue of Weyl's inequality provided by the estimate

$$\lim_{P\to\infty} P^{-2}|f(\lambda_1\alpha)f(\lambda_2\alpha)| = 0, \tag{1.6}$$

valid for α lying on suitable "minor arcs", and subject to the hypothesis that λ_1/λ_2 be irrational. Under such conditions, Freeman's method will establish a formula of the type (1.4) whenever $s > t$. The analysis presented in §4 of this paper instead makes use of an *amplification* procedure that may be loosely described as follows.

We seek to estimate the mean value

$$\int_{\mathfrak{m}} |f(\lambda_1\alpha)\dots f(\lambda_s\alpha)|d\alpha, \tag{1.7}$$

wherein α lies in some set $\mathfrak{m}$ of real numbers, lying in a unit interval, for which the formula (1.6) is known to hold. Suppose that the contribution to this mean value can be adequately controlled whenever $\lambda_i\alpha$ lies in some classical set of minor arcs $\mathfrak{n}$, for some index i, say

$$\int_{\lambda_i^{-1}\mathfrak{n}\cap\mathfrak{m}} |f(\lambda_1\alpha)\dots f(\lambda_s\alpha)|d\alpha = o(P^{s-k}).$$

Then the superior control of the behaviour of $f(\lambda_i\alpha)$ available on the corresponding classical set of major arcs $\mathfrak{N}$ permits an effective application of Hölder's inequality in the form

$$\int_{\lambda_i^{-1}\mathfrak{N}\cap\mathfrak{m}} |f(\lambda_1\alpha)\dots f(\lambda_s\alpha)|d\alpha \leqslant \left(\int_{\lambda_i\alpha\in\mathfrak{N}} |f(\lambda_i\alpha)|^r d\alpha\right)^{1/r} \left(\int_{\mathfrak{m}} \prod_{\substack{j=1\\ j\neq i}}^{s} |f(\lambda_j\alpha)|^{r/(r-1)} d\alpha\right)^{1-1/r}, \tag{1.8}$$

wherein r is a parameter close to, but exceeding, the number $k+1$. The first integral on the right hand side of (1.8) is $O(P^{r-k})$, and the second has the same shape as (1.7), but now with

$$\frac{r}{r-1}(s-1) > s$$

implicit variables. This amplification of the number of variables offers the possibility of successfully applying (1.5) and (1.6) in the style of Freeman [16]. Of course, if necessary, there is the possibility of further amplification by iterating this procedure. The ideas underlying this amplification procedure will not be unfamiliar to experts[1].

Quite apart from avoiding the technical obstructions presented in certain applications of Freeman's method, the above amplification process is also of use in related applications of the Davenport-Heilbronn method and its variants. Let $F(k)$ denote the least integer t with the property that, whenever $s \geqslant t$, one has *for all* large numbers P the asymptotic lower bound

$$N_\tau(P; s, k; \boldsymbol{\lambda}) \gg \tau\Omega(s, k; \boldsymbol{\lambda})P^{s-k},$$

wherein the implicit constant depends at most on s and k. In sections 8, 9 and 10 we establish the conclusions summarised in the following theorem.

Theorem 1.2. *Let $\mathfrak{F}(k)$ denote the integer recorded in the table below. Then for $3 \leqslant k \leqslant 20$ one has $F(k) \leqslant \mathfrak{F}(k)$. Furthermore, when k is large one has*

$$F(k) \leqslant k(\log k + \log\log k + 2 + o(1)).$$

k	3	4	5	6	7	8	9	10	11	12
$\mathfrak{F}(k)$	7	12	18	25	33	42	50	59	67	76

k	13	14	15	16	17	18	19	20
$\mathfrak{F}(k)$	84	92	100	109	117	125	134	142

For large k the conclusion of Theorem 1.2 is contained, albeit in a less explicit and slightly less precise form, in Theorem 1 of Freeman [16]. For smaller values of k our conclusions are new, though less precise versions could be established for $k \geqslant 7$ by combining Freeman's methods with the latest estimates for mean values of smooth Weyl sums made available through the work of Vaughan and Wooley [30]. For smaller k, the aforementioned amplification techniques would appear to play a crucial role in obtaining sharp conclusions. We note also that the upper bounds for $F(k)$ recorded in Theorem 1.2 are, with two exceptions, also the best known upper bounds for the familiar function $G(k)$ in Waring's problem. The two

[1] Jörg Brüdern has kindly pointed out to me that the germs of such ideas occur already in the paper: R. C. Vaughan, *Diophantine approximation by prime numbers, II*, Proc. London Math. Soc. (3) 28 (1974), 385–401.

exceptional cases are $k = 5$ and $k = 6$, where one knows that $G(5) \leqslant 17$ (see Vaughan and Wooley [29]) and $G(6) \leqslant 24$ (see Vaughan and Wooley [28]). The underlying difficulty, in these cases, might be summarised as the absence of an analogue within the Davenport-Heilbronn method of a p-adic iteration restricted to minor arcs only. A precise formula for $\mathfrak{F}(k)$, given in terms of any available mean value estimate for smooth Weyl sums, may be found in section 10 below.

As with Freeman's methods in general, analogues of our conclusions will be easily derived for systems of diophantine inequalities (see Brüdern and Cook [7] and Parsell [20] for more on this topic), and for problems involving prime numbers (see Parsell [21]). As has been remarked by Freeman [16], by adjusting the kernel function employed in the argument, results analogous to those described above can be obtained in problems relevant to the value distribution of diagonal forms. For example, consider positive real numbers $\lambda_1 \dots, \lambda_s$. Then whenever M is sufficiently large in terms of s, k and the positive number τ, one can obtain the expected asymptotic formula for the number of solutions of the inequality

$$|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k - M| < \tau,$$

with $x_i \in \mathbb{N}$ $(1 \leqslant i \leqslant s)$, subject only to the condition that s be accessible to the exponent k. An analogous lower bound may be established, mutatis mutandis, for $s \geqslant \mathfrak{F}(k)$. In particular, provided that the coefficients λ_i $(1 \leqslant i \leqslant s)$ are not all in rational ratio, then the gaps between successive values at integer arguments of the diagonal form $\lambda_1 x_1^k + \cdots + \lambda_s x_s^k$ tend to zero as $|\lambda_1 x_1^k + \cdots + \lambda_s x_s^k| \to \infty$.

We finish by remarking that an alternative approach to the problem of avoiding reference to the convergents of implicit continued fraction expansions occurs already in work of Birch and Davenport [4]. The latter authors were able to make use of work of Cassels [11] concerning the size of integral solutions of quadratic equations. A similar approach has been engineered by Pitman [22] for more general diagonal forms, but only when the number of variables is large. Such an approach is in any case of greater complexity than that of Freeman, motivated in turn by Bentkus and Götze [2], and fails to deliver an asymptotic formula. The author is grateful to Jörg Brüdern and Roger Heath-Brown for conversations on this topic. We should also mention that Eskin, Margulis and Mozes [14] have results on asymptotic formulae for quadratic diophantine inequalities, and that Bentkus and Götze [3] and Freeman [18] have conclusions for certain polynomials of higher degree in a large number of variables.

Our proof of Theorem 1.1 is contained in sections 2–7. We begin in section 2 by preparing upper bound estimates of Weyl type in the style of Freeman. Here we take the liberty of simplifying the argument of Freeman slightly, and also of preparing the estimates in a form suitable to be quoted in any future work. In section 3 we set up Freeman's variant of the Davenport-Heilbronn apparatus. The minor arcs are dismissed in section 4 by use of the amplification technique, and the trivial arcs from the dissection are routinely handled in section 5. We simplify Freeman's analysis of the major arcs in section 6, and then combine the estimates derived in sections 4–6 so as to complete the proof of Theorem 1.1 in section 7. In sections 8, 9 and 10 we move on to discuss the proof of Theorem 1.2, describing the argument for smaller exponents in sections 8 and 9, and larger exponents in section 10.

Throughout, the letter ε will denote a sufficiently small positive number, and P will be a large real number. We use $\ll$ and $\gg$ to denote Vinogradov's notation. In an effort to simplify our account, whenever ε appears in a statement, we assert that the statement holds for every positive number ε. The "value" of ε may consequently change from statement to statement.

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2. Estimates of Weyl type. We begin by establishing versions of Lemmata 4 and 5 of [16]. Here we follow closely the arguments of Freeman, though we incorporate some simplifications that justify a reasonably complete exposition. We begin with the familiar principle that large Weyl sums yield good diophantine approximations.

Lemma 2.1. *There is a positive number c, depending at most on k, with the following property. Suppose that P is a real number, sufficiently large in terms of k, and suppose that γ is a real number with $P^{-2^{-k}} \leqslant \gamma \leqslant 1$. Then whenever $|f(\alpha)| \geqslant \gamma P$, there necessarily exist integers a and q with*

$$(a,q)=1, \quad 1 \leqslant q \leqslant c\gamma^{-2k} \quad \textit{and} \quad |q\alpha - a| \leqslant c\gamma^{-2k}P^{-k}.$$

Proof. The conclusion of the lemma is an "ε-free" version of similar conclusions that may be found, for example, in Chapters 3 and 5 of the book by R. Baker [1]. Suppose that α is a real number with $|f(\alpha)| \geqslant \gamma P$, wherein γ satisfies the hypothesis of the lemma. By Dirichlet's approximation theorem, there exist integers a and q with $(a,q)=1$, $1 \leqslant q \leqslant P^{k-1}$ and $|q\alpha - a| \leqslant P^{1-k}$. If one were to have $q > P$, then it would follow from Weyl's inequality (see Lemma 2.4 of [26]) that

$$|f(\alpha)| \ll P^{1-2^{1-k}+\varepsilon},$$

and for large enough values of P this inequality yields the upper bound

$$|f(\alpha)| < \tfrac{1}{2}P^{1-2^{-k}} \leqslant \tfrac{1}{2}\gamma P.$$

This contradicts our initial hypothesis, and thus we may suppose that $q \leqslant P$. In such circumstances, it follows from Theorem 4.1 and Lemma 4.6 of [26] that

$$|f(\alpha)| \ll P(q+P^k|q\alpha - a|)^{-1/k} + P^{2/3} \ll P(q+P^k|q\alpha - a|)^{-1/(2k)}.$$

Writing c_0 for the implicit constant in the latter inequality, it follows from our initial hypothesis that

$$\gamma P \leqslant |f(\alpha)| \leqslant c_0 P(q+P^k|q\alpha - a|)^{-1/(2k)},$$

whence $q+P^k|q\alpha-a|\leqslant (c_0/\gamma)^{2k}$. The conclusion of the lemma, with $c=c_0^{2k}$, is now immediate.

We next examine products of exponential sums whose arguments are not in rational ratio. In this context it is convenient to introduce non-zero real numbers μ_1 and μ_2 with $\mu_1/\mu_2\notin\mathbb{Q}$. Also, we introduce the notation

$$f_i(\alpha;Q)=\sum_{1\leqslant x\leqslant Q} e(\mu_i\alpha x^k)\quad (i=1,2).$$

Lemma 2.2. *Suppose that S and T are fixed real numbers with $0<S\leqslant 1\leqslant T$. Then one has*

$$\lim_{P\to\infty}\sup_{S\leqslant|\alpha|\leqslant T}\left(P^{-2}|f_1(\alpha;P)f_2(\alpha;P)|\right)=0.$$

Proof. If the desired conclusion fails, then we can find a positive number ε, a sequence of positive real numbers $\{P_n\}$ tending monotonically to infinity, and a corresponding sequence of real numbers $\{\alpha_n\}$ with $\alpha_n\in[S,T]$ $(n\geqslant 1)$, such that for each natural number n one has

$$|f_1(\alpha_n;P_n)f_2(\alpha_n;P_n)|\geqslant \varepsilon P_n^2. \tag{2.1}$$

The trivial estimates $|f_i(\alpha_n;P_n)|\leqslant P_n$ $(i=1,2)$ lead from (2.1) to the lower bounds $|f_i(\alpha_n;P_n)|\geqslant\varepsilon P_n$, valid for $i=1$ and 2. Whenever n is large enough that $P_n\geqslant\varepsilon^{-2^k}$, we may apply Lemma 2.1 with $\gamma=\varepsilon$ so as to infer that for $i=1$ and 2, there exist integers a_{in} and q_{in} with

$$(a_{in},q_{in})=1,\quad 1\leqslant q_{in}\leqslant c\varepsilon^{-2k}\quad\text{and}\quad |\mu_i\alpha_n q_{in}-a_{in}|\leqslant c\varepsilon^{-2k}P_n^{-k}. \tag{2.2}$$

It follows that there are only finitely many possible choices for q_{in}, and the same conclusion holds also for a_{in}, since for large enough n an application of the triangle inequality within (2.2) leads to the upper bound

$$|a_{in}|\leqslant|\mu_i|Tq_{in}+c\varepsilon^{-2k}P_n^{-k}\ll 1.$$

In particular, there are only finitely many choices for the 4-tuple $(a_{1n},q_{1n},a_{2n},q_{2n})$, so that some 4-tuple must occur infinitely often, say (a_1,q_1,a_2,q_2). But by eliminating α_n between the inequalities (2.2) for $i=1$ and 2, one finds that

$$\left|\frac{\mu_1}{\mu_2}-\frac{a_{1n}q_{2n}}{a_{2n}q_{1n}}\right|\ll P_n^{-k}\to 0\quad\text{as}\quad n\to\infty.$$

We therefore conclude that

$$\frac{\mu_1}{\mu_2}=\frac{a_1q_2}{a_2q_1}\in\mathbb{Q},$$

contradicting our hypothesis that μ_1/μ_2 is irrational. This contradiction establishes the conclusion of the lemma.

We now reach the point at which our estimate of Weyl type may be announced.

Lemma 2.3. *Suppose that $S(P)$ is an increasing function of P satisfying $2 \leqslant S(P) \leqslant P$, and such that $S(P) \to \infty$ as $P \to \infty$. Suppose also that μ_1 and μ_2 are non-zero real numbers with $\mu_1/\mu_2 \notin \mathbb{Q}$. Then there exists a function $T(P)$, depending only on μ_1, μ_2 and $S(P)$, with the property that $T(P)$ increases monotonically to infinity with $T(P) \leqslant S(P)$, and such that*

$$\sup_{S(P)P^{-k} \leqslant |\alpha| \leqslant T(P)} |f_1(\alpha;P)f_2(\alpha;P)| \leqslant P^2 T(P)^{-2^{-k-1}}.$$

Proof. For every positive integer m, it follows from Lemma 2.2 that there is a positive real number P_m such that whenever $P \geqslant P_m$ and $1/m \leqslant |\alpha| \leqslant m$, then one has

$$P^{-2}|f_1(\alpha;P)f_2(\alpha;P)| \leqslant 1/m.$$

We may plainly assume, without loss of generality, that the sequence $\{P_m\}$ is non-decreasing, and that $S(P_m) \geqslant m$ for each m. We define the function $T(P)$ by taking $T(P) = m$ for $P_m \leqslant P < P_{m+1}$. It is evident that the function $T(P)$ depends at most on μ_1 and μ_2, that $T(P) \leqslant S(P)$ for each P, and that whenever $P \geqslant P_m$ and $T(P)^{-1} \leqslant |\alpha| \leqslant T(P)$, then

$$P^{-2}|f_1(\alpha;P)f_2(\alpha;P)| \leqslant 1/m.$$

Thus we find that

$$\sup_{T(P)^{-1} \leqslant |\alpha| \leqslant T(P)} |f_1(\alpha;P)f_2(\alpha;P)| \leqslant P^2 T(P)^{-1}. \tag{2.3}$$

Suppose next that $S(P)P^{-k} \leqslant |\alpha| < T(P)^{-1}$, and that

$$|f_1(\alpha;P)| \geqslant T(P)^{-2^{-k-1}} P.$$

Since $T(P) \leqslant P$, we may apply Lemma 2.1 with $\gamma = T(P)^{-2^{-k-1}}$. Thus we deduce that there exist integers a and q with $(a,q) = 1$,

$$1 \leqslant q \leqslant c\gamma^{-2k} \ll T(P)^{1/2} \quad \text{and} \quad |\mu_1 q\alpha - a| \leqslant c\gamma^{-2k} P^{-k} \ll T(P)^{1/2} P^{-k}. \tag{2.4}$$

An application of the triangle inequality within (2.4) leads to the conclusion that

$$|a| \leqslant |\mu_1 \alpha| q + O(P^{-1}) \ll T(P)^{-1/2} \to 0 \quad \text{as} \quad P \to \infty,$$

whence a is necessarily zero for large enough P. The second estimate of (2.4) therefore shows that for large enough P one has

$$|\alpha| < T(P)P^{-k} \leqslant S(P)P^{-k},$$

contradicting our hypothesis that in fact $|\alpha| \geqslant S(P)P^{-k}$. We therefore conclude that

$$\sup_{S(P)P^{-k} \leqslant |\alpha| < T(P)^{-1}} |f_1(\alpha;P)f_2(\alpha;P)| \leqslant P^2 T(P)^{-2^{-k-1}},$$

and in combination with (2.3), this suffices to complete the proof of the lemma.

3. The Davenport-Heilbronn method. It is possible even at this stage to describe the key elements of our application of the Davenport-Heilbronn method. Let s and k be natural numbers with $k \geqslant 2$ and s accessible to k. Also, let $S_i(P)$ $(i = 1, 2)$ denote any functions associated with the accessibility of s to k via the formula (1.3). We consider non-zero real numbers $\lambda_1, \ldots, \lambda_s$, not all in rational ratio, and fix a positive number τ. We now seek to estimate the number $N(P) = N_\tau(P; s, k; \boldsymbol{\lambda})$ of integral solutions of the inequality (1.1) with $\mathbf{x} \in [1, P]^s$. Observe first that when $\lambda_1, \ldots, \lambda_s$ are all of the same sign, then $N(P)$ is finite. There is therefore no loss of generality in supposing that $\lambda_1, \ldots, \lambda_s$ are not all of the same sign, and by relabelling variables, a familiar argument permits the assumption that $\lambda_1/\lambda_2 < 0$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}$. Consider next a function $T(P)$, increasing monotonically to infinity with $T(P) \leqslant S_2(P)$, and growing sufficiently slowly in terms of λ_1 and λ_2 in the context of the conclusion of Lemma 2.3 (as applied with (λ_1, λ_2) in place of (μ_1, μ_2)). We define a function $L(P)$, growing even more slowly than $T(P)$, by putting $L(P) = \max\{1, \log(T(P))\}$.

Before proceeding further, we need to define a kernel function. Here we make use of the work of section 2 of [16].

Lemma 3.1. *Let a and b be real numbers with $0 < a < b$. Then there is an even real function $K(\alpha) = K(\alpha; a, b)$ of the real variable α, such that the function $\psi(\theta)$, defined by*

$$\psi(\theta) = \int_{-\infty}^{\infty} e(\theta\alpha)K(\alpha)d\alpha,$$

satisfies the property that $0 \leqslant \psi(\theta) \leqslant 1$ for all real numbers θ, and moreover

$$\psi(\theta) = \begin{cases} 0, & \text{when } |\theta| \geqslant b, \\ 1, & \text{when } |\theta| \leqslant a. \end{cases}$$

Furthermore, the function K satisfies the bound

$$K(\alpha) \ll \min\{b,\, |\alpha|^{-1},\, (b-a)^{-1}|\alpha|^{-2}\}.$$

Proof. This is the case $h = 1$ of Lemma 1 of [16, section 2.1].

Making use of Lemma 3.1, we define the kernel functions

$$K_-(\alpha) = K(\alpha; \tau(1 - L(P)^{-1}), \tau)$$

and

$$K_+(\alpha) = K(\alpha; \tau, \tau(1 + L(P)^{-1})).$$

Thus, defining the indicator function

$$U_\tau(\theta) = \begin{cases} 0, & \text{when } |\theta| \geqslant \tau, \\ 1, & \text{when } |\theta| < \tau, \end{cases} \tag{3.1}$$

we see that

$$\int_{-\infty}^{\infty} e(\theta\alpha)K_-(\alpha)d\alpha \leqslant U_\tau(\theta) \leqslant \int_{-\infty}^{\infty} e(\theta\alpha)K_+(\alpha)d\alpha. \tag{3.2}$$

Moreover, the expression

$$\Bigl|\int_{-\infty}^{\infty} e(\theta\alpha)K_\pm(\alpha)d\alpha - U_\tau(\theta)\Bigr| \tag{3.3}$$

is zero, except possibly when $||\theta| - \tau| \leqslant \tau L(P)^{-1}$. In the latter circumstances, the expression (3.3) is nonetheless at most 1. We note for future reference at this point that

$$K_\pm(\alpha) \ll_\tau \min\{1,\, |\alpha|^{-1},\, L(P)|\alpha|^{-2}\}. \tag{3.4}$$

Next write

$$f_i(\alpha) = f(\lambda_i\alpha) \quad (1 \leqslant i \leqslant s),$$

and define

$$R_\pm(P) = \int_{-\infty}^{\infty} f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)d\alpha. \tag{3.5}$$

Then it follows from (3.1) and (3.2) that

$$R_-(P) \leqslant N(P) \leqslant R_+(P). \tag{3.6}$$

We aim to obtain asymptotic formulae for $R_-(P)$ and $R_+(P)$ that are asymptotically equal, and thereby we obtain the desired asymptotic formula for $N(P)$.

We divide the real line into three subsets, as is customary in the Davenport-Heilbronn method. The major arc

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \leqslant S_1(P)P^{-k}\}$$

provides the leading term in the ultimate asymptotic formula, while the minor arcs

$$\mathfrak{m} = \{\alpha \in \mathbb{R} : S_1(P)P^{-k} < |\alpha| \leqslant T(P)\},$$

and trivial arcs

$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| > T(P)\}$$

provide contributions asymptotically negligible. We discuss the respective contributions of these sets of arcs in the next three sections.

4. The minor arc contribution. Our treatment of the minor arc contribution, wherein we implicitly apply the Hardy-Littlewood method itself, makes use of the amplification procedure sketched in the introduction. In this context, it is convenient to write $\mathfrak{n} = \mathfrak{n}(S_1(P))$ and $\mathfrak{N} = \mathfrak{N}(S_1(P))$. We begin by observing that the methods of Chapter 4 of [26] (see especially Lemma 4.9 and Theorem 4.4 of [26]) show that whenever $t > \max\{4,\, k+1\}$, one has

$$\int_{\mathfrak{N}} |f(\alpha)|^t d\alpha \ll_t P^{t-k}. \tag{4.1}$$

Since we may suppose that $s > 2k$, it follows from (1.3) and (4.1) that whenever n is a real number and $1 \leqslant i \leqslant s$, one has

$$\begin{aligned}\int_n^{n+1} |f(\lambda_i\alpha)|^s d\alpha &\ll \int_0^1 |f(\beta)|^s d\beta \\ &= \int_{\mathfrak{n}} |f(\beta)|^s d\beta + \int_{\mathfrak{N}} |f(\beta)|^s d\beta \ll P^{s-k}.\end{aligned} \tag{4.2}$$

Next define $\mathfrak{p}$ to be the set of real numbers α with the property that $\lambda_1\alpha \pmod 1$ lies in $\mathfrak{n}$. It is apparent that the set $\mathfrak{P} = \mathbb{R} \setminus \mathfrak{p}$ is equal to the set of real numbers α with the property that $\lambda_1\alpha \pmod 1$ lies in $\mathfrak{N}$. We now observe that the hypothesised bound (1.3) implies that for every real number n, one has

$$\int_{[n,n+1]\cap\mathfrak{p}} |f(\lambda_1\alpha)|^s d\alpha \ll \int_{\mathfrak{n}} |f(\beta)|^s d\beta \ll P^{s-k}(T(P))^{-1}. \tag{4.3}$$

An application of Hölder's inequality consequently leads to the upper bound

$$\int_{[n,n+1]\cap\mathfrak{p}} |f_1(\alpha)\dots f_s(\alpha)| d\alpha \leqslant \prod_{j=1}^{s} I_j^{1/s},$$

where we write

$$I_1 = \int_{[n,n+1]\cap\mathfrak{p}} |f(\lambda_1\alpha)|^s d\alpha \quad \text{and} \quad I_j = \int_n^{n+1} |f(\lambda_j\alpha)|^s d\alpha \quad (2 \leqslant j \leqslant s).$$

Thus we deduce from (4.2) and (4.3) that

$$\int_{[n,n+1]\cap\mathfrak{p}} |f_1(\alpha)\dots f_s(\alpha)| d\alpha \ll P^{s-k}T(P)^{-1/s} \ll P^{s-k}L(P)^{-2}. \tag{4.4}$$

We turn our attention next to the corresponding major arcs $\mathfrak{P}$. We suppose now that $[n, n+1]$ is any interval contained in $\mathfrak{m}$, whence by Lemma 2.3 one has

$$\sup_{\alpha\in[n,n+1]} |f_1(\alpha)f_2(\alpha)| \leqslant P^2 T(P)^{-2^{-k-1}}. \tag{4.5}$$

Recalling again that $s > 2k$, we put $\delta = (s-2k)/2$, and note that

$$s^2/(s+2\delta) = 2(k+\delta)^2/(k+2\delta) > 2k. \tag{4.6}$$

An application of Hölder's inequality provides the bound

$$\int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \leqslant \Bigl(\sup_{\alpha\in[n,n+1]} |f_1(\alpha)f_2(\alpha)|\Bigr)^{\delta/(s+\delta)} J_1^{(s+2\delta)/(s(s+\delta))} \times J_2^{1/(s+\delta)} \prod_{j=3}^{s} J_j^{1/s}, \tag{4.7}$$

where we write

$$J_1 = \int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha)|^{s^2/(s+2\delta)} d\alpha \quad \text{and} \quad J_j = \int_n^{n+1} |f_j(\alpha)|^s d\alpha \quad (2 \leqslant j \leqslant s).$$

In view of (4.6), we deduce from (4.1) that

$$J_1 \ll \int_{\mathfrak{N}} |f(\beta)|^{s^2/(s+2\delta)} d\beta \ll P^{s^2/(s+2\delta)-k}, \tag{4.8}$$

and likewise one finds from (4.2) that for $2 \leqslant j \leqslant s$ one has

$$J_j \ll P^{s-k}. \tag{4.9}$$

Then on substituting (4.5), (4.8) and (4.9) into (4.7), we conclude that

$$\int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \ll P^{s-k}T(P)^{-2^{-k-1}\delta/(s+\delta)} \ll P^{s-k}L(P)^{-2}. \tag{4.10}$$

On combining the estimates (4.4) and (4.10), we find that for every real number n for which $(n, n+1) \subseteq \mathfrak{m}$, one has

$$\int_n^{n+1} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \ll P^{s-k}L(P)^{-2}.$$

In view of the upper bound (3.4) for the kernel function, therefore, it follows that

$$\begin{aligned}\int_{\mathfrak{m}} |f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)|d\alpha &\ll_\tau \int_{S_1(P)P^{-k}}^{1+S_1(P)P^{-k}} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \\ &\quad + \sum_{1\leqslant n\leqslant T(P)} n^{-1} \int_n^{n+1} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \\ &\ll_\tau (1+\log(T(P)))P^{s-k}L(P)^{-2}. \end{aligned} \tag{4.11}$$

We may summarise the discussion of this section in the form of the following lemma.

Lemma 4.1. *One has*

$$\int_{\mathfrak{m}} |f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}L(P)^{-1}.$$

Proof. This is immediate from (4.11), on recalling the definition of $L(P)$.

5. The contribution of the trivial arcs. As is to be expected in applications of the Davenport-Heilbronn method, the disposal of the trivial arcs is routine. An application of Hölder's inequality in combination with (4.2) shows that for all real numbers n, one has

$$\int_n^{n+1} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \leqslant \prod_{i=1}^{s}\left(\int_n^{n+1} |f_i(\alpha)|^s d\alpha\right)^{1/s} \ll P^{s-k}. \tag{5.1}$$

Then on recalling (3.4), we find that

$$\begin{aligned}\int_{\mathfrak{t}} &|f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)|d\alpha \\ &\ll L(P)\sum_{n=0}^{\infty}(n+T(P))^{-2}\int_{n+T(P)}^{n+1+T(P)} |f_1(\alpha)\dots f_s(\alpha)|d\alpha \\ &\ll P^{s-k}L(P)T(P)^{-1} \ll P^{s-k}L(P)^{-1}.\end{aligned}$$

We again summarise the latter conclusion in the form of a lemma.

Lemma 5.1. *One has*

$$\int_{\mathfrak{t}} |f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}L(P)^{-1}.$$

6. The contribution of the major arc. The analysis of section 2.4 of [16] suffices, in principle, to establish an asymptotic formula for the contribution of the major arc within (3.5). Since we are able to make some simplifications in this treatment, and the formulation of our conclusion is in any case somewhat different from that of Freeman, we indulge in a relatively complete exposition.

We begin by replacing the generating functions $f_j(\alpha)$ by their approximations $v_j(\alpha)$, which we define for $1 \leqslant j \leqslant s$ by

$$v_j(\alpha) = \int_0^P e(\lambda_j\alpha\gamma^k)d\gamma.$$

For this purpose, we apply Theorem 4.1 of [26] with $a=0$ and $q=1$ to show that

$$f_j(\alpha) - v_j(\alpha) \ll (1+P^k|\alpha|)^{1/2} \ll S_1(P)^{1/2}, \tag{6.1}$$

uniformly for $\alpha \in \mathfrak{M}$ and $1 \leqslant j \leqslant s$. In addition, we note that the estimate

$$v_j(\alpha) \ll P(1+P^k|\alpha|)^{-1/k} \quad (1 \leqslant j \leqslant s) \tag{6.2}$$

follows by applying integration by parts. On recalling our hypothesis that $S_1(P) \leqslant P$, therefore, we deduce from (6.1) and (6.2) that whenever $\alpha \in \mathfrak{M}$, one has

$$f_j(\alpha) \ll P(1+P^k|\alpha|)^{-1/k} \quad (1 \leqslant j \leqslant s). \tag{6.3}$$

Now s is presumed to be an accessible exponent, so that $s > 2k$, and thus we deduce from (6.1), (6.2) and (6.3) that whenever $\alpha \in \mathfrak{M}$, one has

$$\begin{aligned} f_1(\alpha)\dots f_s(\alpha) - v_1(\alpha)\dots v_s(\alpha) &\ll S_1(P)^{1/2}P^{s-1}(1+P^k|\alpha|)^{-(s-1)/k} \\ &\ll P^{s-1/2}(1+P^k|\alpha|)^{-3/2}. \end{aligned}$$

But $|K_\pm(\alpha)| \ll 1$ uniformly for $\alpha \in \mathfrak{M}$, and hence we may conclude that

$$\begin{aligned} \int_{\mathfrak{M}} f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)d\alpha - &\int_{\mathfrak{M}} v_1(\alpha)\dots v_s(\alpha)K_\pm(\alpha)d\alpha \\ &\ll \int_{-\infty}^{\infty} P^{s-1/2}(1+P^k|\alpha|)^{-3/2}d\alpha \ll P^{s-k-1/2}. \end{aligned} \tag{6.4}$$

Next, again making use of (6.3), it is apparent that the completed singular integral

$$\mathcal{I}_\pm(P) = \int_{-\infty}^{\infty} v_1(\alpha)\dots v_s(\alpha)K_\pm(\alpha)d\alpha$$

converges absolutely, and moreover that

$$\begin{aligned} \int_{\mathbb{R}\setminus\mathfrak{M}} v_1(\alpha)\dots v_s(\alpha)K_\pm(\alpha)d\alpha &\ll \int_{|\alpha|>S_1(P)P^{-k}} P^s(1+P^k|\alpha|)^{-2}d\alpha \\ &\ll P^{s-k}S_1(P)^{-1}. \end{aligned}$$

On combining the latter conclusion with (6.4), we may conclude thus far that

$$\int_{\mathfrak{M}} f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)d\alpha - \mathcal{I}_\pm(P) \ll P^{s-k}L(P)^{-1}. \tag{6.5}$$

In view of the decay of $K_\pm(\alpha)$, moreover, it follows from Fubini's theorem that we may rewrite this singular integral in the shape

$$\mathcal{I}_\pm(P) = \int_0^P \cdots \int_0^P \int_{-\infty}^{\infty} e(\alpha(\lambda_1\gamma_1^k + \dots + \lambda_s\gamma_s^k))K_\pm(\alpha)d\alpha d\boldsymbol{\gamma}. \tag{6.6}$$

The most transparent approach to analysing the singular integral $\mathcal{I}_\pm(P)$ is to linearise by the change of variables $u_i = \mu_i\gamma_i^kP^{-k}$, where we write $\mu_i = |\lambda_i|$ and

$\sigma_i = \lambda_i/\mu_i$ for $1 \leqslant i \leqslant s$, just as in the preamble to the statement of Theorem 1.1. In this way, we deduce from (6.6) that

$$\mathcal{I}_\pm(P) = k^{-s}|\lambda_1 \dots \lambda_s|^{-1/k} P^s \int_{\mathcal{B}_0} (u_1 \dots u_s)^{1/k-1} \Delta_\pm(P;\mathbf{u}) d\mathbf{u}, \tag{6.7}$$

where we write $\mathcal{B}_0$ for the box $[0,\mu_1] \times \dots \times [0,\mu_s]$, and

$$\Delta_\pm(P;\mathbf{u}) = \int_{-\infty}^{\infty} e(\alpha P^k(\sigma_1 u_1 + \dots + \sigma_s u_s)) K_\pm(\alpha) d\alpha.$$

Put

$$\Delta^*(P;\mathbf{u}) = \begin{cases} 0, & \text{when } |\sigma_1 u_1 + \dots + \sigma_s u_s| \geqslant \tau P^{-k}, \\ 1, & \text{when } |\sigma_1 u_1 + \dots + \sigma_s u_s| < \tau P^{-k}. \end{cases} \tag{6.8}$$

Then in view of the discussion of section 3 leading to (3.3), one finds that

$$\Delta_\pm(P;\mathbf{u}) = \Delta^*(P;\mathbf{u}),$$

except possibly when

$$|P^k|\sigma_1 u_1 + \dots + \sigma_s u_s| - \tau| \leqslant \tau L(P)^{-1}, \tag{6.9}$$

in which case one has $|\Delta_\pm(P;\mathbf{u}) - \Delta^*(P;\mathbf{u})| \leqslant 1$. But it is apparent that the measure of the set of points $\mathbf{u} \in \mathcal{B}_0$ that satisfy (6.9) is $O(\tau P^{-k} L(P)^{-1})$. We therefore deduce that

$$\left| \int_{\mathcal{B}_0} (u_1 \dots u_s)^{1/k-1} (\Delta_\pm(P;\mathbf{u}) - \Delta^*(P;\mathbf{u})) d\mathbf{u} \right| \ll \tau P^{-k} L(P)^{-1}. \tag{6.10}$$

Here we note that the contribution to this integral arising from the box $[0, P^{-k}]^s$ is trivially $O(P^{-s})$, and so we may confine our attention to those values of $\mathbf{u}$ for which

$$\tau L(P)^{-1} P^{-k} \Big(\max_{1 \leqslant i \leqslant s} u_i \Big)^{-1} = o(1).$$

Next we observe that our hypothesis $s > 2k$ leads from (6.8), via a simple volume computation, to the estimate

$$\int_{\mathcal{B}_0} (u_1 \dots u_s)^{1/k-1} \Delta^*(P;\mathbf{u}) d\mathbf{u} = 2\tau P^{-k} \left(\int_{\mathcal{S}} (u_1 \dots u_s)^{1/k-1} d\mathcal{S} + O(\tau P^{-k}) \right), \tag{6.11}$$

where $\mathcal{S}$ denotes the set of points $\mathbf{u}$ in $\mathcal{B}_0$ satisfying the equation

$$\sigma_1 u_1 + \dots + \sigma_s u_s = 0.$$

But the integral on the right hand side of (6.11) is equal to the number $C(s,k;\boldsymbol{\lambda})$ defined in the preamble to the statement of Theorem 1.1, and so it follows from (6.10) that

$$\int_{\mathcal{B}_0}(u_1\dots u_s)^{1/k-1}\Delta_\pm(P;\mathbf{u})d\mathbf{u}=2\tau P^{-k}C(s,k;\boldsymbol{\lambda})+O(\tau P^{-k}L(P)^{-1}).$$

On substituting the latter estimate into (6.7), and recalling the definition of the coefficient $\Omega(s,k;\boldsymbol{\lambda})$ from the preamble to the statement of Theorem 1.1, we arrive at the relation

$$\mathcal{I}_\pm(P)=2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k}+O(\tau P^{s-k}L(P)^{-1}).\tag{6.12}$$

We summarise the discussion of this section in the following lemma.

Lemma 6.1. *One has*

$$\int_{\mathfrak{M}}f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)d\alpha=2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k}+O(P^{s-k}L(P)^{-1}).$$

Proof. We merely substitute (6.12) into (6.5), and the conclusion of the lemma is immediate.

7. The proof of Theorem 1.1 and its corollary. The principal conclusions of sections 4, 5 and 6 are easily assembled to complete the proof of Theorem 1.1. First, from (3.5) we note that

$$\left|R_\pm(P)-\int_{\mathfrak{M}}f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)d\alpha\right|\leqslant\int_{\mathfrak{m}\cup\mathfrak{t}}|f_1(\alpha)\dots f_s(\alpha)K_\pm(\alpha)|d\alpha,$$

whence it follows from Lemmata 4.1, 5.1 and 6.1 that

$$|R_\pm(P)-2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k}|\ll P^{s-k}L(P)^{-1}.$$

Next we deduce from (3.6) that the latter estimate yields the relations

$$N(P)\geqslant 2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k}+O(P^{s-k}L(P)^{-1})$$

and

$$N(P)\leqslant 2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k}+O(P^{s-k}L(P)^{-1}),$$

and so the desired asymptotic formula

$$N(P)=2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k}+o(P^{s-k})$$

follows immediately. This completes the proof of Theorem 1.1.

Turning our attention next to the corollary to Theorem 1.1, we begin by considering the situation in which $k \geqslant 3$ and $t \geqslant 2^k$. Here one finds that the estimate (1.3) holds with

$$S_1(P) = P^{k2^{1-k}} \quad \text{and} \quad S_2(P) = \log P.$$

In order to justify this assertion, we note that the conclusion of Lemma F at the end of section 4 of Boklan [5], in combination with the main theorem of that paper, yields the desired conclusion whenever $S_2(P) \ll (\log P)^{3-\varepsilon}$ (the earlier celebrated work of Vaughan [23] on this topic would yield a conclusion only slightly weaker than that which we seek). Next, we recall that Theorem A of Vaughan [24] already establishes (1.3) whenever $S_2(P) \ll (\log P)^{2-\varepsilon}$. Thus, whenever $k \geqslant 3$ and $s \geqslant 2^k$, it follows that the integer s is accessible to the exponent k, and the asymptotic formula (1.4) follows immediately from Theorem 1.1. This completes the proof of the first assertion of the corollary.

Next suppose that $k \geqslant 6$ and $t \geqslant \frac{7}{8}2^k$. Here one may establish the estimate (1.3) with

$$S_1(P) = P \quad \text{and} \quad S_2(P) = (\log P)^{1/2}.$$

In this instance, the desired estimate follows by combining the conclusions of equations (6.6), (8.4), (8.5), and the displayed equation preceding (10.3) of Boklan [6]. The latter work establishes, in fact, a conclusion of the desired type whenever $k \geqslant 6$ and $S_2(P) \leqslant (\log P)^{3/5}$, with larger functions $S_2(P)$ valid for larger exponents k (see Heath-Brown [19] for earlier, less precise, conclusions). It follows, in particular, that whenever $k \geqslant 6$ and $s \geqslant \frac{7}{8}2^k$, then the integer s is accessible to the exponent k, whence the asymptotic formula (1.4) again follows from Theorem 1.1. This completes the proof of the second assertion of the corollary.

Suppose, finally, that k is a large integer. In such circumstances, one may employ the version of Vinogradov's mean value theorem due to Wooley [32] together with Theorem 1 of Ford [15], in combination with any suitable variant of Vinogradov's version of Weyl's inequality (see, for example, Theorem 5.3 of [26]), to show that (1.3) holds with

$$t = k^2(\log k + \log\log k + 8), \quad S_1(P) = P/(2k), \quad S_2(P) = P^{1/(5\log k)}.$$

An account of such a conclusion may be found, for example, in the discussion of Brüdern, Kawada and Wooley [8] leading to equation (4.25) of the latter paper. Thus we find that whenever

$$s \geqslant \lceil k^2(\log k + \log\log k + 8)\rceil, \tag{7.1}$$

and k is large, then the integer s is accessible to the exponent k, and the asymptotic formula (1.4) follows from Theorem 1.1. This completes the proof of the final assertion of the corollary.

We finish this section by noting that, as will be anticipated, the lower order terms in (7.1) are certainly susceptible to improvement. Indeed, for $k \geqslant 9$ or thereabouts, numerical work associated with Vinogradov's mean value theorem leads to rather sharper bounds than are available either from (7.1) or indeed the first conclusions of the corollary (see Ford [15]; there is also sharper unpublished work of Boklan and Wooley on this topic).

8. Asymptotic lower bounds: smaller exponents, I. The proof of Theorem 1.2 can be modelled largely on that of Theorem 1.1, although the use of smooth numbers leads to several complications. In particular, for smaller exponents, one must employ both smooth Weyl sums and classical Weyl sums within the attendant application of the Davenport-Heilbronn method. We begin with an analogue of Lemma 2.3 applicable for smooth Weyl sums. In this context, we define the set of R-smooth numbers up to P by

$$\mathcal{A}(P,R)=\{n\in[1,P]\cap\mathbb{Z}\,:\,p\text{ prime and }p|n\text{ implies }p\leqslant R\},$$

and the corresponding smooth Weyl sum $h(\alpha)=h_k(\alpha;P,R)$ by

$$h_k(\alpha;P,R)=\sum_{x\in\mathcal{A}(P,R)}e(\alpha x^k).$$

In our applications here we take $R=P^\eta$ with η a sufficiently small positive number, and with this in mind it is occasionally convenient to write $h(\alpha;P)=h_k(\alpha;P,P^\eta)$.

Lemma 8.1. *Suppose that k is an integer with $k\geqslant 3$, and that $S(P)$ is an increasing function of P satisfying $2\leqslant S(P)\leqslant P$, and such that $S(P)\to\infty$ as $P\to\infty$. Suppose also that μ_1 and μ_2 are non-zero real numbers with $\mu_1/\mu_2\notin\mathbb{Q}$. Then there exists a positive number $A=A(k)$, and a function $T(P)$ depending only on μ_1, μ_2 and $S(P)$, with the property that $T(P)$ increases monotonically to infinity with $T(P)\leqslant S(P)$, and such that*

$$\sup_{S(P)P^{-k}\leqslant|\alpha|\leqslant T(P)}|h(\mu_1\alpha;P)h(\mu_2\alpha;P)|\leqslant P^2T(P)^{-A(k)}.$$

Proof. We begin by observing that an analogue of Lemma 2.1 holds. Namely, there are positive numbers $B=B(k)$ and $c=c(k,\eta)$ with the following property. Suppose that P is a real number, sufficiently large in terms of k and η, and suppose that γ is a real number with $P^{-B}\leqslant\gamma\leqslant 1$. Then whenever $|h(\alpha)|\geqslant\gamma P$, there necessarily exist integers a and q with

$$(a,q)=1,\quad 1\leqslant q\leqslant c\gamma^{-3k}\quad\text{and}\quad|q\alpha-a|\leqslant c\gamma^{-3k}P^{-k}.$$

In order to justify this assertion, we apply the argument of the proof of Lemma 2.1, but we pause en route in order to recall some of the literature familiar to aficionados of the modern circle method.

Suppose that α is a real number with $|h(\alpha)|\geqslant\gamma P$, wherein γ satisfies the hypotheses of the lemma. By Dirichlet's approximation theorem, there exist integers a and q with $(a,q)=1$, $1\leqslant q\leqslant P^{k-1/2}$ and $|q\alpha-a|\leqslant P^{1/2-k}$. If one were to have $q>P^{1/2}$, then it would follow from Lemma 3.1 of Wooley [34] that

$$|h(\alpha)|\ll P^{1-\sigma(k)+\varepsilon},$$

where $\sigma(k) = 4^{-k}$. Here we have opted for a weak but cheap consequence of the latter lemma wherein we put $\lambda = \frac{1}{2} + \varepsilon$ and $t = w = 2^{k-1}$, so that Hua's lemma provides the permissible exponents $\Delta_t = \Delta_w = 0$. Then provided that P is sufficiently large, and $B(k)$ is at most 5^{-k}, say, we find that

$$|h(\alpha)| < \tfrac{1}{2}P^{1-B} \leqslant \tfrac{1}{2}\gamma P,$$

and this contradicts our hypothesis that $|h(\alpha)| \geqslant \gamma P$. We are therefore forced to conclude that $q \leqslant P^{1/2}$.

We now apply Lemmata 7.2 and 8.5 of Vaughan and Wooley [27] to deduce that

$$|h(\alpha)| \ll P(q + P^k|q\alpha - a|)^{-1/(3k)}. \tag{8.1}$$

The first of the latter lemmata, applied with $M = P^{1/2+\varepsilon}$, implies that for $q \leqslant P^{1/2}$, one has

$$|h(\alpha)| \ll (\log P)^3 q^\varepsilon P(q + P^k|q\alpha - a|)^{-1/(2k)} + P^{7/8+\varepsilon}, \tag{8.2}$$

and the right hand side of (8.2) is majorised by that of (8.1) whenever

$$q + P^k|q\alpha - a| \geqslant (\log P)^{100k}.$$

When $1 \leqslant q \leqslant (\log P)^{100k}$ and $|q\alpha - a| \leqslant (\log P)^{100k}P^{-k}$, on the other hand, the second of the aforementioned lemmata establishes that

$$|h(\alpha)| \ll q^\varepsilon P(q + P^k|q\alpha - a|)^{-1/k} + P(\log P)^{-100k}, \tag{8.3}$$

and the right hand side of (8.3) is majorised by that of (8.1) whenever

$$q + P^k|q\alpha - a| \leqslant (\log P)^{100k}.$$

Writing c_0 for the implicit constant in (8.1), it follows from our hypothesis $|h(\alpha)| \geqslant \gamma P$ that

$$\gamma P \leqslant |h(\alpha)| \leqslant c_0 P(q + P^k|q\alpha - a|)^{-1/(3k)},$$

whence

$$q + P^k|q\alpha - a| \leqslant (c_0/\gamma)^{3k}.$$

The desired analogue of Lemma 2.1, with $B(k) = 5^{-k}$ and $c = c_0^{3k}$, now follows immediately.

It is now possible to establish an analogue of Lemma 2.2 to the effect that whenever S and T are fixed real numbers with $0 < S \leqslant 1 \leqslant T$, then

$$\lim_{P\to\infty} \sup_{S\leqslant|\alpha|\leqslant T} \left(P^{-2}|h(\mu_1\alpha; P)h(\mu_2\alpha; P)|\right) = 0.$$

The argument of the proof of Lemma 2.2 applies with obvious changes that need not detain us here. From this point we may follow the argument of the proof of

Lemma 2.3, again with only cosmetic alterations, and thereby the conclusion of the present lemma follows with $A(k) = 5^{-k}$. Needless to say, refinement in this permissible value of $A(k)$ is certainly feasible, but this apparently has only invisible consequences.

Next we launch our application of the Davenport-Heilbronn method. We suppose for the moment that k is an integer with $3 \leqslant k \leqslant 6$, and we put $s = \mathfrak{F}(k)$, where $\mathfrak{F}(k)$ denotes the integer defined in the table occurring in the statement of Theorem 1.2. We consider non-zero real numbers $\lambda_1, \dots, \lambda_s$, not all in rational ratio, and a fixed positive number τ. Finally, we take ν and η to be sufficiently small positive numbers, and put $R = P^\eta$ and $S(P) = (\log P)^\nu$. We now seek to estimate the number $N^*(P)$ of integral solutions of the inequality (1.1) with $1 \leqslant x_1 \leqslant P$ and $x_j \in \mathcal{A}(P,R)$ $(2 \leqslant j \leqslant s)$. Observe again that when $\lambda_1, \dots, \lambda_s$ are all of the same sign, then $N^*(P)$ is finite. The familiar argument therefore permits us to assume that $\lambda_2/\lambda_3 < 0$ and $\lambda_2/\lambda_3 \notin \mathbb{Q}$. Next we consider a function $T(P)$, increasing monotonically to infinity with $T(P) \leqslant S(P)$, and growing sufficiently slowly in terms of λ_2 and λ_3 in the context of the conclusion of Lemma 8.1 (applied with (λ_2, λ_3) in place of (μ_1, μ_2)). We put $L(P) = \max\{1, \log(T(P))\}$, and define the kernel functions $K_\pm(\alpha)$ as in section 3. Finally, we recall the definition of $f_i(\alpha)$ from section 3, write $h_i(\alpha) = h(\lambda_i \alpha)$ $(1 \leqslant i \leqslant s)$, and define

$$R^*_\pm(P) = \int_{-\infty}^{\infty} f_1(\alpha) h_2(\alpha) \dots h_s(\alpha) K_\pm(\alpha) d\alpha. \tag{8.4}$$

As before, we seek to establish asymptotic formulae for $R^*_\pm(P)$, whence the relation

$$R^*_-(P) \leqslant N^*(P) \leqslant R^*_+(P) \tag{8.5}$$

leads to an asymptotic formula for $N^*(P)$.

On this occasion our division of the real line into three subsets goes as follows. The major arc $\mathfrak{M}$ is defined by

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \leqslant S(P) P^{-k}\},$$

the minor arcs $\mathfrak{m}$ are defined by

$$\mathfrak{m} = \{\alpha \in \mathbb{R} : S(P) P^{-k} < |\alpha| \leqslant T(P)\},$$

and we put

$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| > T(P)\}.$$

We estimate the contribution within (8.4) arising from each of these subsets in the next section.

9. Asymptotic lower bounds: smaller exponents, II. The amplification procedure described in section 4 again plays a critical role in our analysis of the minor arcs in the Davenport-Heilbronn method, but now additional preparation is required in order to handle the inhomogeneous set-up embodied in (8.4). As a first step in this direction, we recall a number of mean value estimates from the literature.

Lemma 9.1. *When $3 \leqslant k \leqslant 6$, let $u = u(k)$, $v = v(k)$ and $\lambda = \lambda_u(k)$ denote the exponents recorded in the table below. Suppose that η is sufficiently small in terms of k. Then one has*

$$\int_0^1 |h_k(\alpha; P, P^\eta)|^u d\alpha \ll P^\lambda, \tag{9.1}$$

and whenever $w \geqslant v(k)$, one has

$$\int_0^1 |h_k(\alpha; P, P^\eta)|^w d\alpha \ll P^{w-k}. \tag{9.2}$$

k	3	4	5	6
$u(k)$	6	11	17	24
$v(k)$	7.7	12	18	26
$\lambda(k)$	3.2495	7.1068	12.0387	18.0001

Proof. We have taken the liberty of recording the sharpest available exponents in the statement of the lemma, although earlier, weaker, conclusions would suffice for our purposes. We begin by discussing cubic smooth Weyl sums. Here, the upper bound (9.1) is immediate from Theorem 1.2 of Wooley [35], while the estimate (9.2) with $w \geqslant 7.7$ follows from Theorem 2 of Brüdern and Wooley [10] together with the trivial estimate $|h(\alpha)| \leqslant P$.

Next consider the situation in which $k = 4$. Here the estimate (9.2) with $w \geqslant 12$ follows from the trivial estimate $|h(\alpha)| \leqslant P$ together with Lemma 5.2 of Vaughan [25], on considering the underlying diophantine equation (see, for example, the proof of Lemma 2.3 of Brüdern and Wooley [9]). But from the table in section 2 of [9], or by considering the underlying diophantine equation and making use of Theorem 2 of the latter paper, one finds that

$$\int_0^1 |h_4(\alpha; P, P^\eta)|^{10} d\alpha \ll P^{6.213431}.$$

Consequently, on recalling the estimate (9.2) with $w = 12$, an application of Schwarz's inequality yields the upper bound

$$\begin{aligned}\int_0^1 |h_4(\alpha; P, P^\eta)|^{11} d\alpha &\ll \left(\int_0^1 |h_4(\alpha; P, P^\eta)|^{10} d\alpha\right)^{1/2} \left(\int_0^1 |h_4(\alpha; P, P^\eta)|^{12} d\alpha\right)^{1/2} \\ &\ll P^\lambda,\end{aligned}$$

where

$$\lambda = (6.213431 + 8)/2 < 7.1068.$$

Suppose next that $k = 5$. Then the estimate (9.2) with $w \geqslant 18$ follows from the trivial estimate $|h(\alpha)| \leqslant P$ together with Lemma 7.3 of Vaughan and Wooley [29]. Meanwhile, from the table in the appendix to the latter paper, one finds that

$$\int_0^1 |h_5(\alpha; P, P^\eta)|^{16} d\alpha \ll P^{11.077363}.$$

In combination with the estimate (9.2), an application of Schwarz's inequality therefore reveals that

$$\int_0^1 |h_5(\alpha;P,P^\eta)|^{17}d\alpha \ll \left(\int_0^1 |h_5(\alpha;P,P^\eta)|^{16}d\alpha\right)^{1/2}\left(\int_0^1 |h_5(\alpha;P,P^\eta)|^{18}d\alpha\right)^{1/2}$$
$$\ll P^\lambda,$$

where

$$\lambda = (11.077363+13)/2 < 12.0387.$$

Finally, when $k=6$, again making use of the trivial estimate $|h(\alpha)| \leqslant P$, the upper bound (9.2) with $w \geqslant 26$ is essentially immediate from the discussion following the proof of Lemma 7.1 of Vaughan and Wooley [28], and follows easily from the argument of the proof of Lemma 7.3 of Vaughan and Wooley [29]. The estimate (9.1) with $u=24$ and $\lambda = 18+\varepsilon$, on the other hand, is immediate from the tables in the appendix to [29].

This completes the proof of the lemma.

We are now equipped to discuss the minor arc contribution, but pause briefly in order to introduce further notation and to recall some well known estimates for classical Weyl sums. We write now $\mathfrak{n} = \mathfrak{n}(P)$ and $\mathfrak{N} = \mathfrak{N}(P)$, and note that the methods of Chapter 4 of [26] again establish the estimate (4.1) for $t > k+1$. Throughout this discussion we drop explicit mention of k in notation, since context will ensure clarity. Also, in this section and the next, we write $\mathfrak{F}(k)$ to denote the integer recorded in the table appearing in the statement of Theorem 1.2.

Lemma 9.2. *Whenever $s \geqslant \mathfrak{F}(k)$, one has*

$$\int_{\mathfrak{m}} |f_1(\alpha)h_2(\alpha)h_3(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}L(P)^{-1}.$$

Proof. It follows from Weyl's inequality (see, for example, Lemma 2.4 of [26]) that

$$\sup_{\alpha\in\mathfrak{n}} |f(\alpha)| \ll P^{1-2^{1-k}+\varepsilon}. \tag{9.3}$$

Define $\mathfrak{p}$ to be the set of real numbers α with the property that $\lambda_1\alpha \pmod 1$ lies in $\mathfrak{n}$. Then, just as in section 4, we see that the set $\mathfrak{P} = \mathbb{R}\setminus\mathfrak{p}$ is equal to the set of real numbers α with the property that $\lambda_1\alpha \pmod 1$ lies in $\mathfrak{N}$. But by Hölder's inequality combined with the trivial estimate $|h_j(\alpha)| \leqslant P$, one finds that for every real number n,

$$\int_{[n,n+1]\cap\mathfrak{p}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)|d\alpha$$
$$\leqslant \Big(\sup_{\alpha\in\mathfrak{p}}|f(\lambda_1\alpha)|\Big)\int_n^{n+1}|h_2(\alpha)\dots h_s(\alpha)|d\alpha$$
$$\leqslant P^{s-1-u}\Big(\sup_{\beta\in\mathfrak{n}}|f(\beta)|\Big)\prod_{i=2}^s\left(\int_n^{n+1}|h(\lambda_i\alpha)|^u d\alpha\right)^{1/(s-1)}.$$

Then in view of (9.3) and the conclusion of Lemma 9.1, we deduce that for $3 \leqslant k \leqslant 6$ one has

$$\int_{[n,n+1]\cap\mathfrak{p}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)|d\alpha \ll P^{s-u-2^{1-k}+\varepsilon}\int_0^1 |h(\beta)|^u d\beta \ll P^{s-k-\delta}, \tag{9.4}$$

for some positive number $\delta \geqslant 0.0004$.

Next we consider the corresponding set $\mathfrak{P}$. Suppose that $[n,n+1]$ is any interval contained in $\mathfrak{m}$, so that by Lemma 8.1 there is a positive number A for which

$$\sup_{\alpha\in[n,n+1]} |h_2(\alpha)h_3(\alpha)| \leqslant P^2T(P)^{-A}. \tag{9.5}$$

Define $\theta = \theta(k,s)$ by

$$\theta = \left(\frac{2k+1}{2k+3} - \frac{2v^2}{v^3+1}\right)\frac{1}{s-3}.$$

Then on combining the trivial estimate $|h_j(\alpha)| \leqslant P$ with an application of Hölder's inequality, one finds that

$$\int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)|d\alpha \leqslant P^{(s-3)(1-v\theta)}\Big(\sup_{\alpha\in[n,n+1]} |h_2(\alpha)h_3(\alpha)|\Big)^{1/(v^3+1)} \times K_1^{2/(2k+3)}(K_2K_3)^{v^2/(v^3+1)}\prod_{j=4}^s K_j^\theta, \tag{9.6}$$

where we write

$$K_1 = \int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha)|^{k+3/2}d\alpha \quad\text{and}\quad K_j = \int_n^{n+1} |h_j(\alpha)|^v d\alpha \quad (2 \leqslant j \leqslant s).$$

In order to confirm the validity of the application of Hölder's inequality underlying (9.6), one has only to check that for $3 \leqslant k \leqslant 6$ one has $v(k)\theta(k,s) < 1$. But by hypothesis, one has $s \geqslant \mathfrak{F}(k)$, and so a modest computation confirms the desired inequality in all cases under consideration. In view of (4.1) and the estimates (9.2) of Lemma 9.1, one has

$$K_1 \ll P^{3/2} \quad\text{and}\quad K_j \ll P^{v-k} \quad (2 \leqslant j \leqslant s).$$

Consequently, on substituting these estimates together with (9.5) into (9.6), we arrive at the upper bound

$$\int_{[n,n+1]\cap\mathfrak{P}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)|d\alpha \ll P^{s-k}T(P)^{-A/(v^3+1)} \ll P^{s-k}L(P)^{-2}. \tag{9.7}$$

On combining (9.4) and (9.7), we find that for every real number n for which $(n, n+1) \subseteq \mathfrak{m}$, one has

$$\int_n^{n+1} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)|d\alpha \ll P^{s-k}L(P)^{-2}. \tag{9.8}$$

Then on exploiting the decay of the kernel function, just as in the derivation of (4.11), we obtain

$$\int_{\mathfrak{m}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha \ll (1+\log(T(P)))P^{s-k}L(P)^{-2},$$

and the conclusion of the lemma is now immediate.

The trivial arcs are easily decimated, as we now see.

Lemma 9.3. *Whenever $s \geqslant \mathfrak{F}(k)$, one has*

$$\int_{\mathfrak{t}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}L(P)^{-1}.$$

Proof. On replacing the estimate (9.5) by the trivial bound $|h_2(\alpha)h_3(\alpha)| \leqslant P^2$, we find that the argument of the proof of Lemma 9.2 leading to (9.8), via (9.4) and (9.7), now yields the upper bound

$$\int_n^{n+1} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}, \tag{9.9}$$

uniformly for $n \in \mathbb{R}$. On substituting (9.9) for (5.1) in the argument of section 5, we now conclude that

$$\int_{\mathfrak{t}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}L(P)T(P)^{-1} \ll P^{s-k}L(P)^{-1}.$$

This completes the proof of the lemma.

In order to treat the major arc $\mathfrak{M}$, we begin by noting that Lemma 8.5 of Wooley [31] (see also Lemma 5.4 of Vaughan [25] for a related conclusion) shows that there exists a positive number $c = c(\eta)$ such that

$$\sup_{\alpha \in \mathfrak{M}} |h_j(\alpha) - cv_j(\alpha)| \ll P(\log P)^{-1/2} \ll PS(P)^{-10}. \tag{9.10}$$

On recalling (6.1) and (6.2), therefore, we deduce that whenever $\alpha \in \mathfrak{M}$, one has

$$f_1(\alpha)h_2(\alpha)\dots h_s(\alpha) - c^{s-1}v_1(\alpha)v_2(\alpha)\dots v_s(\alpha) \ll P^sS(P)^{-10}.$$

But $|K_\pm(\alpha)| \ll 1$ uniformly for $\alpha \in \mathfrak{M}$, and $\mathfrak{M}$ has measure $O(S(P)P^{-k})$. Write

$$\mathcal{K}_\pm = \int_{\mathfrak{M}} v_1(\alpha)\dots v_s(\alpha)K_\pm(\alpha)d\alpha.$$

Then we conclude that

$$\int_{\mathfrak{M}} f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)d\alpha - c^{s-1}\mathcal{K}_\pm \ll P^{s-k}S(P)^{-9}. \tag{9.11}$$

Thus, on noting that the argument leading from (6.4) to the conclusion of Lemma 6.1 provides the estimate

$$\mathcal{K}_\pm - 2\tau\Omega(s,k;\boldsymbol{\lambda})P^{s-k} \ll P^{s-k}L(P)^{-1}, \tag{9.12}$$

we may conclude as follows.

Lemma 9.4. *Whenever $s > 2k$, one has*

$$\int_{\mathfrak{M}} f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)d\alpha = 2\tau c^{s-1}\Omega(s,k;\boldsymbol{\lambda})P^{s-k} + O(P^{s-k}L(P)^{-1}).$$

Proof. This is immediate from (9.11) and (9.12).

We now complete the proof of Theorem 1.2 for $3 \leqslant k \leqslant 6$ by noting that from (8.4) one has

$$\left| R_\pm^*(P) - \int_{\mathfrak{M}} f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)d\alpha \right| \leqslant \int_{\mathfrak{m}\cup\mathfrak{t}} |f_1(\alpha)h_2(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha.$$

Then whenever $s \geqslant \mathfrak{F}(k)$, it follows from Lemmata 9.2, 9.3 and 9.4 that

$$|R_\pm^*(P) - 2\tau c^{s-1}\Omega(s,k;\boldsymbol{\lambda})P^{s-k}| \ll P^{s-k}L(P)^{-1}.$$

In view of (8.5), one therefore obtains the asymptotic formula

$$N^*(P) = 2\tau c^{s-1}\Omega(s,k;\boldsymbol{\lambda}) + o(P^{s-k})$$

that is, in fact, more explicit than the asymptotic lower bound claimed in the statement of Theorem 1.2.

10. Asymptotic lower bounds: larger exponents. The bulk of the work required to prove Theorem 1.2, for the cases in which $k \geqslant 7$, has already been accomplished in sections 8 and 9, so we may be brief in our discussion at this point. However, it seems appropriate to discuss the analysis in a fairly general setting in order that future applications may be more easily executed, and this requires some additional notation. We say that an exponent $\Delta_s = \Delta_{s,k}$ is *admissible* whenever the exponent has the property that, whenever $R \leqslant P^\eta$ with $\eta > 0$ sufficiently small in terms of s and k, one has

$$\int_0^1 |h(\alpha; P, P^\eta)|^{2s} d\alpha \ll P^{\lambda_{s,k}+\varepsilon}$$

with $\lambda_{s,k} = 2s - k + \Delta_{s,k}$. Tables of exponents $\lambda_{s,k}$ associated with such admissible exponents may be found in the work of Vaughan and Wooley [29], [30]. For larger values of k, one may apply the following result of Wooley [33].

Lemma 10.1. *Let $k \geqslant 4$ and $t \in \mathbb{N}$. For each $s \in \mathbb{N}$ with $2 \leqslant s \leqslant t$, define the real number $\Delta_s = \Delta_{s,k}$ to be the unique positive solution of the equation*

$$\Delta_s e^{\Delta_s/k} = ke^{1-2s/k}.$$

Then $\Delta_s = \Delta_{s,k}$ is an admissible exponent, and hence the exponent $\Delta^*_{s,k} = ke^{1-2s/k}$ is also admissible.

Proof. This is the corollary to Theorem 2.1 of [33].

Associated to the admissible exponents $\Delta_{t,k}$ $(t \geqslant k)$ is a Weyl exponent $\sigma(k)$. Let s, t and w be natural numbers satisfying $2s \geqslant k + 1$, and suppose that Δ_n $(n = s, t, w)$ are admissible exponents. Define

$$\sigma(k) = \frac{k - \Delta_t - \Delta_s\Delta_w}{2(s(k + \Delta_w - \Delta_t) + tw(1 + \Delta_s))},$$

and

$$\lambda(k) = \frac{s(k - \Delta_t) + tw\Delta_s}{s(k + \Delta_w - \Delta_t) + tw(1 + \Delta_s)}.$$

Then Corollary 1 to Theorem 4.2 of Wooley [34] shows that whenever $1/2 < \lambda(k) < 1 - \sigma(k)$ and $\alpha \in \mathfrak{n}(P)$, then one has

$$|h_k(\alpha; P, P^\eta)| \ll P^{1-\sigma(k)+\varepsilon}. \tag{10.1}$$

In Corollary 2 to Theorem 4.2 of [34] it is shown, inter alia, that such Weyl exponents exist with $\sigma(k)^{-1} = k(\log k + O(\log \log k))$.

We now launch our application of the Davenport-Heilbronn method. We suppose that k is an integer with $k \geqslant 4$, that Δ_n $(n \geqslant 3)$ are admissible exponents, and that $\sigma(k)$ is an associated Weyl exponent. Let t be any integer with

$$t > \min_{\substack{v \geqslant 2k+1 \\ v \in \mathbb{N}}} (2v + \Delta_v/\sigma^*(k)), \tag{10.2}$$

where $\sigma^*(k) = \min\{\sigma(k), 1/8\}$. We aim to show that the number $F(k)$, defined in the preamble to Theorem 1.2, satisfies the upper bound $F(k) \leqslant t$. The upper bounds for $F(k)$ when $7 \leqslant k \leqslant 20$ then follow immediately from the tables of Vaughan and Wooley [30], and the corresponding upper bound for large k follows from the discussion of section 5 of Wooley [34].

Consider again non-zero real numbers $\lambda_1, \dots, \lambda_s$, not all in rational ratio, and fix a positive number τ. We now seek to estimate the number $\widetilde{N}(P)$ of integral solutions of the inequality (1.1) with $x_j \in \mathcal{A}(P, P^\eta)$ $(1 \leqslant j \leqslant s)$. Just as before, there is no loss of generality in supposing that $\lambda_2/\lambda_3 < 0$ and $\lambda_2/\lambda_3 \notin \mathbb{Q}$. We introduce the functions $T(P)$ and $L(P)$, as in section 8, and define the kernel functions $K_\pm(\alpha)$ also as in section 8. On this occasion, we define

$$\widetilde{R}_\pm(P) = \int_{-\infty}^{\infty} h_1(\alpha) \dots h_s(\alpha) K_\pm(\alpha) d\alpha, \tag{10.3}$$

and we note as before that

$$\widetilde{R}_-(P) \leqslant \widetilde{N}(P) \leqslant \widetilde{R}_+(P). \tag{10.4}$$

Finally, our division into major, minor and trivial arcs is that described in section 8.

We begin with an analogue of Lemma 9.1 that yields useful mean value estimates.

Lemma 10.2. *Suppose that t is any real number satisfying* (10.2). *Then one has*

$$\int_0^1 |h_k(\alpha; P, P^\eta)|^t d\alpha \ll_t P^{t-k}.$$

Proof. Put $\mathfrak{N} = \mathfrak{N}(P)$, $\mathfrak{n} = \mathfrak{n}(P)$, and $h(\alpha) = h_k(\alpha; P, P^\eta)$. Then it follows from (10.1) and (10.2) that

$$\begin{aligned}\int_{\mathfrak{n}} |h(\alpha)|^t d\alpha &\leqslant \Big(\sup_{\alpha\in\mathfrak{n}} |h(\alpha)|\Big)^{t-2v} \int_0^1 |h(\alpha)|^{2v} d\alpha \\ &\ll (P^{1-\sigma(k)+\varepsilon})^{t-2v} P^{2v-k+\Delta_v+\varepsilon} \ll P^{t-k-\phi},\end{aligned} \tag{10.5}$$

for some positive number ϕ. Define the function $H(\alpha)$ for $\alpha \in \mathbb{R}$ by putting

$$H(\alpha) = P(q + P^k|q\alpha - a|)^{-1/(2k+1)} \tag{10.6}$$

when $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N} \pmod 1$, and by setting $H(\alpha)$ to be zero otherwise. Then, as in the discussion of section 9 of Vaughan and Wooley [27], one finds that whenever $\alpha \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}$,

$$|h(\alpha)| \ll H(\alpha) + P^{7/8+\varepsilon}. \tag{10.7}$$

Here we note that Lemma 7.2 of [27] has been applied with $M = P^{3/4}$. In particular, it follows that

$$\int_{\mathfrak{N}} |h(\alpha)|^t d\alpha \ll (P^{1-\sigma(k)+\varepsilon})^{t-2v} \int_0^1 |h(\alpha)|^{2v} d\alpha + \int_{\mathfrak{N}} H(\alpha)^{t-2v} |h(\alpha)|^{2v} d\alpha.$$

Consequently, proceeding just as in the derivation of (10.5), and then applying Hölder's inequality, we deduce that for some positive number ϕ one has

$$\int_{\mathfrak{N}} |h(\alpha)|^t d\alpha \ll P^{t-k-\phi} + \left(\int_{\mathfrak{N}} H(\alpha)^t d\alpha\right)^{1-2v/t} \left(\int_0^1 |h(\alpha)|^t d\alpha\right)^{2v/t}.$$

On substituting from (10.5), we therefore obtain the upper bound

$$\begin{aligned}\int_0^1 |h(\alpha)|^t d\alpha &= \int_{\mathfrak{N}} |h(\alpha)|^t d\alpha + \int_{\mathfrak{n}} |h(\alpha)|^t d\alpha \\ &\ll P^{t-k-\phi} + \left(\int_{\mathfrak{N}} H(\alpha)^t d\alpha\right)^{1-2v/t} \left(\int_0^1 |h(\alpha)|^t d\alpha\right)^{2v/t},\end{aligned}$$

whence

$$\begin{aligned}\int_0^1 |h(\alpha)|^t d\alpha &\ll P^{t-k} + \sum_{1\leqslant q\leqslant P} q^{-1/(5k)} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-\infty}^{\infty} \frac{P^t}{(q+P^k|q\alpha-a|)^2} d\alpha \\ &\ll_t P^{t-k}.\end{aligned}$$

The proof of the lemma is now complete.

The argument required to establish an asymptotic formula for $\widetilde{R}_{\pm}(P)$ is now routine, and contained in most essentials within the work of Freeman [16], [17]. Our sharper analysis here saves at most a variable or two over that potentially available to Freeman. Note first that by combining (10.1) and (10.7), we have the upper bound

$$|h(\alpha)| \ll H(\alpha) + P^{1-\sigma^*(k)+\varepsilon}, \tag{10.8}$$

uniformly for $\alpha \in [0,1]$. Suppose then that s is an integer and t is a real number, and that s and t satisfy the inequalities

$$s > t > \min_{\substack{v\geqslant 2k+1\\ v\in\mathbb{N}}} (2v + \Delta_v/\sigma^*(k)).$$

We put $\delta = \min\{1, s-t\}$. In view of (10.8), for every real number n one has the upper bound

$$\begin{aligned}&\int_n^{n+1} |h_1(\alpha)\dots h_s(\alpha)| d\alpha \\ &\quad\leqslant \left(P^{1-\sigma^*(k)+\varepsilon}\right)^{\delta} \int_n^{n+1} |h(\lambda_1\alpha)|^{1-\delta} |h_2(\alpha)\dots h_s(\alpha)| d\alpha \\ &\qquad + \int_n^{n+1} |H(\lambda_1\alpha)|^{\delta} |h(\lambda_1\alpha)|^{1-\delta} |h_2(\alpha)\dots h_s(\alpha)| d\alpha.\end{aligned} \tag{10.9}$$

Since

$$s-\delta > \min_{\substack{v\geqslant 2k+1\\ v\in\mathbb{N}}} \left(2v+\Delta_v/\sigma^*(k)\right),$$

it follows from Lemma 10.2, via Hölder's inequality, that

$$\int_n^{n+1} |h(\lambda_1\alpha)|^{1-\delta}|h_2(\alpha)\dots h_s(\alpha)|d\alpha \ll \max_{1\leqslant i\leqslant s}\int_n^{n+1} |h(\lambda_i\alpha)|^{s-\delta}d\alpha \ll P^{s-\delta-k}.$$

Then in view of (10.8), one may conclude that

$$\left(P^{1-\sigma^*(k)+\varepsilon}\right)^{\delta}\int_n^{n+1} |h(\lambda_1\alpha)|^{1-\delta}|h_2(\alpha)\dots h_s(\alpha)|d\alpha \ll P^{s-k-\tau}, \tag{10.10}$$

for a positive number τ with $\tau > \frac{1}{2}\delta\sigma^*(k)$.

Suppose next that $(n,n+1)\subseteq\mathfrak{m}$. Then Lemma 8.1 supplies the bound

$$\sup_{\alpha\in[n,n+1]} |h_2(\alpha)h_3(\alpha)| \leqslant P^2T(P)^{-A}, \tag{10.11}$$

for some $A=A(k)>0$. Let ϕ be a sufficiently small positive number. Then an application of Hölder's inequality supplies the bound

$$\begin{aligned}\int_n^{n+1} &|H(\lambda_1\alpha)|^{\delta}|h(\lambda_1\alpha)|^{1-\delta}|h_2(\alpha)\dots h_s(\alpha)|d\alpha\\ &\leqslant \Big(\sup_{\alpha\in[n,n+1]}|h_2(\alpha)\dots h_s(\alpha)|\Big)^{\phi}\left(\int_n^{n+1}|H(\lambda_1\alpha)|^{4k+3}d\alpha\right)^{\delta/(4k+3)}\\ &\quad\times\left(\int_n^{n+1}|h_1(\alpha)|^{\kappa}d\alpha\right)^{(1-\delta)/\kappa}\prod_{i=2}^{s}\left(\int_n^{n+1}|h_i(\alpha)|^{\kappa}d\alpha\right)^{(1-\phi)/\kappa},\end{aligned} \tag{10.12}$$

where we have chosen κ to satisfy

$$((s-1)(1-\phi)+(1-\delta))/\kappa+\delta/(4k+3)=1.$$

That there exists such a choice of κ, with $\kappa>t$, follows from our hypothesis to the effect that $s>t>4k+3$. But since $\kappa>t$, Lemma 10.2 implies that

$$\int_n^{n+1}|h_i(\alpha)|^{\kappa}d\alpha \ll P^{\kappa-k} \quad (1\leqslant i\leqslant s).$$

Furthermore, it is easily deduced from (10.6) that

$$\int_n^{n+1}|H(\lambda_1\alpha)|^{4k+3}d\alpha \ll P^{3k+3}.$$

Thus we conclude from (10.11) and (10.12) that

$$\int_n^{n+1} |H(\lambda_1\alpha)|^\delta |h(\lambda_1\alpha)|^{1-\delta}|h_2(\alpha)\dots h_s(\alpha)|d\alpha \ll (P^{s-1}T(P)^{-A})^\phi P^{s-(s-1)\phi-k} \ll P^{s-k}T(P)^{-A\phi}. \tag{10.13}$$

Combining (10.9), (10.10) and (10.13), we see thus far that whenever $(n, n+1) \subseteq \mathfrak{m}$, then

$$\int_n^{n+1} |h_1(\alpha)\dots h_s(\alpha)|d\alpha \ll P^{s-k}L(P)^{-2}. \tag{10.14}$$

The upper bound

$$\int_{\mathfrak{m}} |h_1(\alpha)\dots h_s(\alpha)|d\alpha \ll P^{s-k}L(P)^{-1}$$

now follows from (10.14) just as in the corresponding part of the proof of Lemma 9.2. The estimate

$$\int_{\mathfrak{t}} |h_1(\alpha)\dots h_s(\alpha)K_\pm(\alpha)|d\alpha \ll P^{s-k}L(P)^{-1}$$

may now be obtained, as in the proof of Lemma 9.3, from the argument already applied to treat the minor arcs $\mathfrak{m}$. Finally, on making use of (9.10), we may imitate the argument of the proof of Lemma 9.4 so as to obtain the formula

$$\int_{\mathfrak{M}} h_1(\alpha)\dots h_s(\alpha)K_\pm(\alpha)d\alpha = 2\tau c^s\Omega(s,k;\boldsymbol{\lambda})P^{s-k} + o(P^{s-k}L(P)^{-1}).$$

Assembling the above estimates, we find from (10.3) and (10.4) that the upper bound

$$|\widetilde{R}_\pm(P) - 2\tau c^s\Omega(s,k;\boldsymbol{\lambda})P^{s-k}| \ll P^{s-k}L(P)^{-1},$$

and hence

$$\widetilde{N}(P) = 2\tau c^s\Omega(s,k;\boldsymbol{\lambda}) + o(P^{s-k}),$$

now follow just as before. This completes the proof of Theorem 1.2.

We note in finishing that a variant of Lemma 5.4 of Vaughan and Wooley [30] would enable the condition $t \geqslant 2k+1$ in (10.2) to be relaxed to the weaker constraint $t \geqslant [k/2] + 2$, with concommitant improvements in the tabulated values of $\mathfrak{F}(k)$ whenever sufficiently strong admissible exponents are available.

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