ADDITIVE REPRESENTATION IN SHORT INTERVALS, I: WARING'S PROBLEM FOR CUBES

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1. Introduction. Although available technology frequently fails to establish that all large integers are represented in some prescribed additive form, there are many situations in which one is nonetheless able to conclude that almost all such integers are thus represented. When the numbers that we seek to represent lie in some thin subsequence of the integers, the existence of such a conclusion might reasonably be construed as additional evidence in favour of the assertion that all large integers are represented in the given form. Pursuing this line of enquiry, a sequence of papers arising from work of the authors joint with Kawada has been devoted to the investigation of exceptional sets primarily in thin polynomial sequences (see [5], [6], [7], [8]), and also in certain arithmetic sequences such as the set of prime numbers (see [9]). On this occasion, we turn our attention to the problem of establishing strong exceptional set estimates in short intervals, a topic that has already received much attention in the literature (see especially [1], [10], [14], [15], [17] and [18]). Although our ideas in this context are quite widely applicable, we concentrate in this paper on applications to Waring's problem for cubes. The reader will experience no difficulty in generalising our results to analogous problems involving higher powers.

Denote by $E_s(N)$ the number of natural numbers up to N that cannot be written as the sum of s cubes of natural numbers. After work of Linnik [16] and Davenport [11], respectively, it is known that $E_s(N) \ll 1$ for $s \geq 7$, and that $E_s(N) = o(N)$ for $s \geq 4$. Indeed, the most recent developments in the circle method permit one to show that whenever ε is a sufficiently small positive number, then

$$E_4(N) \ll N^{37/42-\varepsilon}, \quad E_5(N) \ll N^{5/7-\varepsilon} \quad \text{and} \quad E_6(N) \ll N^{23/42-\varepsilon} \tag{1.1}$$

(see Brüdern [4] and Wooley [23], [24] for the first of these estimates, and the discussion in §1 of Brüdern, Kawada and Wooley [5] for the second and third estimates). When $4 \leq s \leq 6$, denote by β_s the least non-negative number satisfying

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the property that whenever $\theta > \beta_s$, and η is a sufficiently small positive number, then one has

$$E_s(N+N^\theta) - E_s(N) \ll N^{\theta-\eta}.$$

In view of (1.1), it is of course trivial that $\beta_4 < 37/42$, $\beta_5 < 5/7$ and $\beta_6 < 23/42$.

The first non-trivial results concerning exceptional sets for sums of cubes in short intervals are due to Brüdern and Watt, who established that $\beta_4 \le 3/4$ (see Theorem 2 of [10]). This conclusion was subsequently improved by Kawada [14], who proved that

$$\beta_4 \le 1585/2169 = 0.730751\dots. \tag{1.2}$$

Note that whenever almost all integers in the interval

$$\Big[N - [N^{1/3}]^3, N - [N^{1/3}]^3 + M\Big]$$

are sums of s cubes of natural numbers, then almost all integers in the interval $[N, N+M]$ are sums of $s+1$ cubes of natural numbers. It is therefore apparent that $\beta_s \le \frac{2}{3}\beta_{s-1}$ $(s = 5, 6)$, whence, in particular, Kawada's bound (1.2) leads to the bounds

$$\beta_5 < 0.487168 \quad \text{and} \quad \beta_6 < 0.324779. \tag{1.3}$$

Following the development of an auxiliary mean value estimate in §2, we are able to exploit the ideas underlying papers in our earlier series joint with Kawada in order to establish, in §3, a conclusion that yields a further improvement in the estimate for β_6 provided in (1.3).

Theorem 1.1. *Whenever $17/63 < \theta \le 1$, and η is a sufficiently small positive number, one has*

$$E_6(N+N^\theta) - E_6(N) \ll N^{\theta-\eta}.$$

In particular, one has

$$\beta_6 \le 17/63 = 0.269841\dots.$$

We remark that by incorporating the mean value estimates of Wooley [24] into the arguments of Brüdern [4] underlying Lemma 3.2 of [5], one would obtain the slightly sharper upper bound $\beta_6 < 0.269674$. We are also able to establish a bound for β_5 slightly sharper than that recorded in (1.3). In this instance our improvements are more modest than those embodied in Theorem 1.1, and the associated methods are considerably more complicated. Following the preparation of an auxiliary estimate in §4, we therefore defer the proof of this new bound to §5.

Theorem 1.2. *Whenever $10/21 \le \theta \le 1$, and η is a sufficiently small positive number, one has*

$$E_5(N+N^\theta) - E_5(N) \ll N^{\theta-\eta}.$$

In particular, one has

$$\beta_5 \le 10/21 = 0.476190\dots.$$

We turn our attention next to analogues of the above exceptional set problems in which one seeks to show that the expected asymptotic formula for the number of representations holds almost always. Denote by $R_s(n)$ the number of representations of n as the sum of s cubes of natural numbers. A heuristic application of the circle method suggests that for $s \geq 4$, one should have

$$R_s(n) = \frac{\Gamma(4/3)^s}{\Gamma(s/3)} \mathfrak{S}_s(n) n^{s/3-1} + o(n^{s/3-1}), \tag{1.4}$$

where

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(q^{-1} \sum_{r=1}^{q} e(ar^3/q) \Big)^s e(-na/q), \tag{1.5}$$

and $e(z)$ denotes $e^{2\pi i z}$. It is known that for $s \geq 4$, the singular series $\mathfrak{S}_s(n)$ satisfies the lower bound $\mathfrak{S}_s(n) \gg 1$ (see, for example, Theorem 4.5 of Vaughan [22]), and so the relation (1.4) does indeed constitute an asymptotic formula. In order to measure the frequency with which the expected asymptotic formula (1.4) might fail, when $\psi(\tau)$ is a function of a positive variable τ, we define $\widetilde{E}_s(N;\psi)$ to be the number of integers n with $1 \leq n \leq N$ for which

$$\left| R_s(n) - \frac{\Gamma(4/3)^s}{\Gamma(s/3)} \mathfrak{S}_s(n) n^{s/3-1} \right| > n^{s/3-1} \psi(n)^{-1}. \tag{1.6}$$

As was essentially pointed out in [7], it follows from work of Vaughan [19], as refined by Boklan [2], that whenever $\psi(\tau)$ is an increasing function growing sufficiently slowly, then one has

$$\widetilde{E}_{4+t}(N;\psi) \ll N^{1-t/6} (\log N)^{\varepsilon - 3 + t/2} \psi(N)^2 \quad (0 \leq t \leq 3). \tag{1.7}$$

Moreover, the celebrated work of Vaughan [19] shows, under the same hypotheses on ψ, that $\widetilde{E}_8(N;\psi) \ll 1$. The bound (1.7) has recently been sharpened by Wooley [25] in the case $t = 3$ to obtain

$$\widetilde{E}_7(N;\psi) \ll N^{4/9+\varepsilon}. \tag{1.8}$$

Developing the ideas from our previous work [7], joint with Kawada, concerning asymptotic formulae, we establish in §6 a short intervals analogue of the above bounds for $\widetilde{E}_s(N;\psi)$ for $5 \leq s \leq 7$.

Theorem 1.3. *Suppose that $\psi(\tau)$ is a function of a positive variable τ, increasing monotonically to infinity, and satisfying $\psi(\tau) = O(\tau^{\delta})$ for some sufficiently small positive number δ. Also, let M and N be large positive numbers with $M \leq N$. Then for $1 \leq t \leq 3$, and for each positive number ε, one has*

$$\widetilde{E}_{4+t}(N+M;\psi) - \widetilde{E}_{4+t}(N;\psi) \ll (N^{(5-t)/6} + MN^{(1-t)/6})(\log N)^{\varepsilon-(7-t)/2} \psi(N)^2.$$

Plainly, the conclusion of Theorem 1.3 provides non-trivial estimates for exceptional sets of integers of size N, in short intervals of size

$$\begin{cases} N^{2/3}(\log N)^{\varepsilon-3}, & \text{for sums of 5 cubes,} \\ N^{1/2}(\log N)^{\varepsilon-5/2}, & \text{for sums of 6 cubes,} \\ N^{1/3}(\log N)^{\varepsilon-2}, & \text{for sums of 7 cubes.} \end{cases}$$

For sums of 4 cubes, meanwhile, one has the earlier conclusion of Brüdern and Watt [10] demonstrating that whenever $M = N^\theta$ with $5/6 < \theta < 1$, then one has

$$\widetilde{E}_4(N + M; (\log \tau)^{1/5}) - \widetilde{E}_4(N; (\log \tau)^{1/5}) \ll M(\log N)^{-1/4}.$$

It is evident that both the latter conclusion, and the estimates stemming from Theorem 1.3, go beyond what is trivially available via the upper bounds (1.7) and (1.8). Perhaps it is worth remarking also that the argument used to establish Theorem 1.3 is easily adapted to show that whenever $M = N^\theta$ with $\theta > 1/6$, and δ is a sufficiently small positive number satisfying $2\delta < \theta - 1/6$, then one has

$$\widetilde{E}_8(N + M; \tau^\delta) - \widetilde{E}_8(N; \tau^\delta) \ll MN^{-\delta},$$

a conclusion that goes beyond that automatically available from the aforementioned result of Vaughan concerning sums of eight cubes.

Since the basic plan of attack in such problems is described in detail within our earlier paper [5] joint with Kawada, we avoid discussing details of strategy at this point. It is sufficient to remark that we encode information concerning exceptional integers within an exponential sum, and then exploit this exponential sum explicitly via mean value estimates familiar to those expert in applications of the circle method. This approach retains the local information concerning the set of exceptions that is more difficult to exploit via more traditional approaches involving the use of Bessel's inequality.

Throughout, the letter ε will denote a sufficiently small positive number. We take P to be the basic parameter, a large real number depending at most on ε. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε. Sometimes we make use of vector notation. For example, the expression $(c_1, \ldots, c_t)$ is abbreviated to $\mathbf{c}$. Also we write $[x]$ for the greatest integer not exceeding x. In an effort to simplify our analysis, we adopt the following convention concerning the parameter ε. Whenever ε appears in a statement, we assert that for each $\varepsilon > 0$ the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε.

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2. An auxiliary mean value estimate. We establish in this section a mean value estimate crucial to the strength of Theorem 1.1. Before announcing this bound, we introduce some notation. Let P be a large positive number, and take M to be a real number with $1 \le M \le P^{1/3}$. We then write $Q = PM^{-1}$ and $H = PM^{-3}$. Next let N and Z be large positive numbers, and suppose that $\mathcal{Z}$ is a set of integers with $\mathcal{Z} \subseteq [N, N+Z]$. It is convenient to abbreviate $\mathrm{card}(\mathcal{Z})$ simply to $\widehat{Z}$. Finally, we introduce the exponential sums

$$f(\alpha) = \sum_{P<x\le 2P} e(\alpha x^3), \quad g(\alpha) = \sum_{Q<y\le 2Q} e(\alpha y^3)$$

and

$$K(\alpha) = \sum_{n\in\mathcal{Z}} e(n\alpha).$$

Lemma 2.1. *Suppose that the parameters M, P, Q, Z, and the quantity $\widehat{Z}$, satisfy the inequalities*

$$M^3 \le \widehat{Z} \le Z \le \min\{2P, Q^2\}.$$

Then for each positive number ε, one has

$$\int_0^1 |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha \ll P^\varepsilon \left((P+P^{1/2}H)Q\widehat{Z} + (PQ)^{1/2}(P+H^2)^{1/2}\widehat{Z}^{3/2}\right). \tag{2.1}$$

Proof. On considering the underlying diophantine equation, it follows from orthogonality that the integral on the left hand side of (2.1) is equal to the number of solutions of the diophantine equation

$$x_1^3 - x_2^3 = y_1^3 - y_2^3 + n_1 - n_2, \tag{2.2}$$

with $P < x_i \le 2P$ $(i = 1,2)$, $Q < y_j \le 2Q$ $(j = 1,2)$ and $n_l \in \mathcal{Z}$ $(l = 1,2)$. Given any solution $\mathbf{x}$, $\mathbf{y}$, $\mathbf{n}$ counted by the latter equation, it is evident from our hypotheses on Z that

$$|y_1^3 - y_2^3 + n_1 - n_2| < (2Q)^3 + Z < 9Q^3.$$

Meanwhile, whenever $x_1 \ne x_2$, one has

$$9Q^3 > |x_1^3 - x_2^3| > 3P^2|x_1 - x_2|,$$

and thus $|x_1 - x_2| < 3H$. In the latter situation, on substituting $z = x_1 + x_2$ and $h = x_1 - x_2$, we deduce from (2.2) that

$$h(3z^2 + h^2) = 4(y_1^3 - y_2^3 + n_1 - n_2),$$

wherein $1 \le z \le 4P$ and $1 \le |h| \le 3H$. Write

$$F(\alpha) = \sum_{1\le h\le 3H} \sum_{1\le z\le 4P} e(\alpha h(3z^2 + h^2)).$$

Then on considering the underlying diophantine equations, we conclude thus far that

$$\int_0^1 |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha \ll PI_1 + I_2, \tag{2.3}$$

where

$$I_1 = \int_0^1 |g(\alpha)K(\alpha)|^2 d\alpha \quad \text{and} \quad I_2 = \int_0^1 |F(\alpha)g(4\alpha)^2 K(4\alpha)^2| d\alpha. \tag{2.4}$$

The mean value I_1 is easily estimated. By orthogonality, one finds that I_1 is bounded above by the number of integral solutions of the equation

$$y_1^3 - y_2^3 = n_1 - n_2,$$

with $Q < y_j \le 2Q$ $(j = 1, 2)$ and $n_l \in \mathcal{Z}$ $(l = 1, 2)$. If such a solution were to exist with $y_1 \ne y_2$, then one would have

$$3Q^2 < |y_1^3 - y_2^3| = |n_1 - n_2| \le Z.$$

Since the latter condition contradicts our hypothesis that $Z \le Q^2$, we conclude that, necessarily, one has $y_1 = y_2$ and $n_1 = n_2$, whence

$$I_1 \ll Q\widehat{Z}. \tag{2.5}$$

We estimate I_2 by means of the Hardy-Littlewood method. Define the set of major arcs $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \le PQ^{-3}\},$$

with $0 \le a \le q \le P$ and $(a, q) = 1$. Also, define the corresponding set of minor arcs by $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. The argument of the proof of Lemma 3.1 of Vaughan [21] shows that whenever $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $|\alpha - a/q| \le q^{-2}$, then one has

$$\sum_{1\le h\le 3H} \Big| \sum_{1\le z\le 4P} e(\alpha h(3z^2 + h^2)) \Big|^2$$
$$\ll P^\varepsilon \left(\frac{P^2 H}{q + Q^3 |q\alpha - a|} + PH + q + Q^3 |q\alpha - a| \right).$$

By Cauchy's inequality, therefore, one deduces that under the same hypotheses, one has

$$F(\alpha) \ll P^{\varepsilon} H^{1/2} \left(\frac{P^2 H}{q + Q^3 |q\alpha - a|} + PH + q + Q^3 |q\alpha - a| \right)^{1/2}. \tag{2.6}$$

Suppose that $\alpha \in \mathfrak{m}$. By Dirichlet's approximation theorem, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \le a \le q \le P^{-1}Q^3$, $(a, q) = 1$ and $|q\alpha - a| \le PQ^{-3}$. But $\alpha \notin \mathfrak{M}$, so that necessarily one has $q > P$. We therefore conclude from (2.6) that

$$\sup_{\alpha \in \mathfrak{m}} |F(\alpha)| \ll P^{\varepsilon} H^{1/2} (PH + P^{-1}Q^3 + P)^{1/2} \ll P^{1/2+\varepsilon} H.$$

On applying the latter bound in combination with Schwarz's inequality, and recalling the bound (2.5), we thus deduce from (2.4) that

$$\begin{aligned} I_2 &\le \Big(\sup_{\alpha \in \mathfrak{m}} |F(\alpha)| \Big) I_1 + \int_{\mathfrak{M}} |F(\alpha) g(4\alpha)^2 K(4\alpha)^2| d\alpha \\ &\ll P^{1/2+\varepsilon} HQ\widehat{Z} + I_3^{1/2} I_4^{1/2}, \end{aligned} \tag{2.7}$$

where we write

$$I_3 = \int_0^1 |F(\alpha)^2 K(4\alpha)^2| d\alpha \quad \text{and} \quad I_4 = \int_{\mathfrak{M}} |g(4\alpha)^4 K(4\alpha)^2| d\alpha.$$

By orthogonality, the mean value I_3 is bounded above by the number of integral solutions of the equation

$$h_1(3z_1^2 + h_1^2) - h_2(3z_2^2 + h_2^2) = 4(n_1 - n_2), \tag{2.8}$$

with $1 \le h_i \le 3H$ $(i = 1, 2)$, $1 \le z_j \le 4P$ $(j = 1, 2)$ and $n_l \in \mathcal{Z}$ $(l = 1, 2)$. Let I_5 denote the number of the latter solutions for which

$$h_1^3 - h_2^3 = 4(n_1 - n_2), \tag{2.9}$$

and let I_6 denote the corresponding number of solutions for which (2.9) fails to hold. Consider first any one of the $O(H^2 \widehat{Z}^2)$ possible choices of $\mathbf{h}$ and $\mathbf{n}$ for which (2.9) does not hold. On writing ν for the fixed integer $4(n_1 - n_2) + h_2^3 - h_1^3 \ne 0$, we see from (2.8) that $3h_1 z_1^2 - 3h_2 z_2^2 = \nu$. But then the elementary theory of binary quadratic forms (see, for example, Estermann [12]) shows that the number of possible choices for z_1 and z_2 is $O(P^{\varepsilon})$. Thus we find that

$$I_6 \ll P^{\varepsilon} H^2 \widehat{Z}^2. \tag{2.10}$$

Consider next any one of the $O(\widehat{Z}^2)$ possible choices for $\mathbf{n}$ with $n_1 \ne n_2$. By an elementary divisor function estimate, there are $O(H^{\varepsilon})$ possible choices for $h_1 - h_2$

and $h_1^2+h_1h_2+h_2^2$ satisfying (2.9), whence $O(H^\varepsilon)$ possible choices for h_1 and h_2. Given a fixed such choice of $\mathbf{n}$ and $\mathbf{h}$, the equation (2.8) becomes $h_1z_1^2=h_2z_2^2$, whence there are at most $4P$ possible choices for $\mathbf{z}$. The number of solutions $\mathbf{h}$, $\mathbf{z}$, $\mathbf{n}$ counted by I_5 with $n_1\ne n_2$ is therefore $O(H^\varepsilon P\widehat{Z}^2)$. Plainly, the corresponding number with $n_1=n_2$, and consequently also $h_1=h_2$ and $z_1=z_2$, is $O(PH\widehat{Z})$. Then we conclude that

$$I_5\ll H^\varepsilon P\widehat{Z}^2+PH\widehat{Z},$$

whence, on recalling (2.10), we have

$$I_3\ll P^\varepsilon(H^2+P)\widehat{Z}^2+PH\widehat{Z}. \tag{2.11}$$

It remains to estimate I_4, and this requires some further notation. Write

$$S(q,a)=\sum_{r=1}^q e(ar^3/q)\quad\text{and}\quad v(\beta;L)=\int_L^{2L}e(\beta\gamma^3)d\gamma. \tag{2.12}$$

We then define the function $g^*(\alpha)$ for $\alpha\in[0,1)$ by putting

$$g^*(\alpha)=q^{-1}S(q,4a)v(4(\alpha-a/q);Q),$$

when $\alpha\in\mathfrak{M}(q,a)\subseteq\mathfrak{M}$, and by setting $g^*(\alpha)=0$ otherwise. It follows from Theorem 4.1 of Vaughan [22] that

$$\sup_{\alpha\in\mathfrak{M}}|g(4\alpha)-g^*(\alpha)|\ll P^{1/2+\varepsilon},$$

and hence we deduce that

$$I_4\ll\int_{\mathfrak{M}}|g(4\alpha)g^*(\alpha)K(4\alpha)|^2d\alpha+P^{1+\varepsilon}I_1,$$

where I_1 is defined as in (2.4). Write

$$I_7=\int_{\mathfrak{M}}|g^*(\alpha)^4K(4\alpha)^2|d\alpha.$$

Then on applying Schwarz's inequality, and recalling the bound (2.5), we obtain

$$I_4\ll I_4^{1/2}I_7^{1/2}+P^{1+\varepsilon}Q\widehat{Z},$$

whence

$$I_4\ll I_7+P^{1+\varepsilon}Q\widehat{Z}. \tag{2.13}$$

But the methods of Chapter 4 of Vaughan [22] suffice to establish that

$$I_7\le K(0)^2\int_{\mathfrak{M}}|g^*(\alpha)|^4d\alpha\ll Q^{1+\varepsilon}\widehat{Z}^2,$$

and thus we conclude from (2.13) that

$$I_4\ll P^{1+\varepsilon}Q\widehat{Z}+Q^{1+\varepsilon}\widehat{Z}^2. \tag{2.14}$$

On recalling our hypotheses concerning $\widehat{Z}$, we find from (2.7), (2.11) and (2.14) that

$$I_2\ll P^{1/2+\varepsilon}HQ\widehat{Z}+P^\varepsilon(P+H^2)^{1/2}\widehat{Z}(PQ\widehat{Z})^{1/2}.$$

Finally, the upper bound recorded in the statement of the lemma follows on recalling (2.3) and (2.5).

3. Sums of six cubes in short intervals. Our objective in this section is the proof of Theorem 1.1. The skeleton of our argument here follows closely the pattern established in previous parts of our earlier series of papers joint with Kawada, though we require some preparation in order to bring our analysis to a successful conclusion. We consider a large natural number N and a positive number θ with $17/63 < \theta \le 1$, and we write $Z = N^\theta$. Define next $\mathcal{Z} = \mathcal{Z}(N,Z)$ to be the set of integers n with $N < n \le N + Z$ that cannot be written as the sum of 6 cubes of natural numbers. It is convenient to abbreviate $\text{card}(\mathcal{Z}(N,Z))$ to $\widehat{Z} = \widehat{Z}(N,Z)$. Write $\delta = \frac{1}{2}(\theta - 17/63)$. We claim that the conclusion of Theorem 1.1 follows on demonstrating that whenever $Z \le N^{1/3}$, then one has $\widehat{Z} = O(ZN^{-\delta/2})$. For if $Z > N^{1/3}$, then we may subdivide the interval $(N, N+Z]$ into at most $[ZN^{-1/3}]+1$ subintervals $(N_0, N_0 + N_0^{1/3}]$ of length $N_0^{1/3}$, on each of which we may infer that $\widehat{Z}(N_0, N_0^{1/3}) \ll N_0^{1/3-\delta/2}$. But then we have

$$\widehat{Z}(N,Z) \ll (ZN^{-1/3}+1)N^{1/3-\delta/2} \ll ZN^{-\delta/2},$$

and this establishes our earlier claim. The conclusion of Theorem 1.1 follows on noting that

$$\widehat{Z}(N,Z) = E_6(N+Z) - E_6(N).$$

Henceforth, therefore, we may suppose that $\theta \le 1/3$.

We take $P = \frac{1}{2}N^{1/3}$, $M = P^{1/6}$, and define Q and H as in §2. It follows that

$$Z > N^{17/63} > N^{1/6+2\delta} > M^3 N^\delta,$$

and thus there is no loss of generality in supposing that $\widehat{Z} \ge M^3$, for otherwise one has $\widehat{Z} < M^3 < ZN^{-\delta}$, and this suffices to establish Theorem 1.1 as before. In combination with the discussion of the previous paragraph, we may suppose henceforth that the hypotheses of Lemma 2.1 are satisfied.

In order to make use of recent technology employed in Waring's problem for cubes, we recall some generating functions introduced in Brüdern, Kawada and Wooley [5]. Let η be a sufficiently small positive number depending at most on ε, and consider a real number R with $P^{\eta/2} < R \le P^\eta$. We write

$$S = P^{6/7}, \quad Y = P^{1/7},$$

and define the generating functions

$$\mathfrak{f}_p(\alpha) = \sum_{\substack{P<x\le 2P\\ p\nmid x}} e(\alpha x^3), \quad \mathfrak{g}(\alpha) = \sum_{S<y\le 2S} e(\alpha y^3), \quad \mathfrak{h}(\alpha) = \sum_{z\in\mathcal{A}(S,R)} e(\alpha z^3),$$

where

$$\mathcal{A}(S,R) = \{z \in [1,S]\cap\mathbb{Z} : p|z \text{ and } p \text{ prime} \Rightarrow p \le R\}.$$

Finally, we define the generating function

$$\mathcal{F}(\alpha)=\sum_{\substack{Y<p\le 2Y\\ p\equiv 2\ (\mathrm{mod}\ 3)}}\mathfrak{f}_p(\alpha)\mathfrak{g}(\alpha p^3)\mathfrak{h}(\alpha p^3)^2,$$

where the summation is over prime numbers.

We may now begin our proof of Theorem 1.1 in earnest. Recall the definition of the exponential sums $f(\alpha)$ and $g(\alpha)$ from §2, and write

$$R(n)=\int_0^1 \mathcal{F}(\alpha)f(\alpha)g(\alpha)e(-n\alpha)\,d\alpha. \tag{3.1}$$

Then it is apparent that whenever $n\in\mathcal{Z}$, one has $R(n)=0$. Defining the exponential sum $K(\alpha)$ as in §2, we therefore conclude from (3.1) that

$$\int_0^1 \mathcal{F}(\alpha)f(\alpha)g(\alpha)K(-\alpha)\,d\alpha=\sum_{n\in\mathcal{Z}}R(n)=0. \tag{3.2}$$

We interpret (3.2) by means of the Hardy-Littlewood method. Write $L=(\log P)^{1/100}$, and define $\mathfrak{P}$ to be the union of the intervals

$$\mathfrak{P}(q,a)=\{\alpha\in[0,1)\,:\,|q\alpha-a|\le LP^{-3}\},$$

with $0\le a\le q\le L$ and $(a,q)=1$. We then denote the corresponding set of minor arcs by $\mathfrak{p}=[0,1)\setminus\mathfrak{P}$. We have the following lower bound for the contribution of the major arcs $\mathfrak{P}$ to the integral $R(n)$.

Lemma 3.1. *Suppose that $N<n\le N+Z$. Then one has*

$$\int_{\mathfrak{P}}\mathcal{F}(\alpha)f(\alpha)g(\alpha)e(-n\alpha)\,d\alpha\gg M^{-1}YS^3(\log Y)^{-1}.$$

Proof. By following the argument of the proof of Lemma 2.1 of [5], one finds without difficulty that for $N<n\le N+Z$, one has

$$\begin{aligned}\int_{\mathfrak{P}}\mathcal{F}(\alpha)f(\alpha)g(\alpha)e(-n\alpha)\,d\alpha&\gg P^{-3}\mathcal{F}(0)f(0)g(0)\\&\gg P^{-3}(YPS^3(\log Y)^{-1})PQ.\end{aligned}$$

The presence of the shortened exponential sum $g(\alpha)$ in place of a longer one causes no difficulties in the implicit analysis. The conclusion of the lemma consequently follows on recalling that $Q=PM^{-1}$.

Employing the lower bound provided by Lemma 3.1 together with the definition of $K(\alpha)$, we see that

$$\begin{aligned}\int_{\mathfrak{P}} \mathcal{F}(\alpha)f(\alpha)g(\alpha)K(-\alpha)d\alpha &\gg \sum_{n\in\mathcal{Z}} M^{-1}YS^3(\log Y)^{-1}\\ &= \widehat{Z}M^{-1}YS^3(\log Y)^{-1}.\end{aligned}$$

On substituting the latter bound into (3.2), we conclude thus far that

$$\Big|\int_{\mathfrak{p}} \mathcal{F}(\alpha)f(\alpha)g(\alpha)K(-\alpha)d\alpha\Big| \gg \widehat{Z}M^{-1}YS^3(\log Y)^{-1}. \tag{3.3}$$

We now aim to obtain an upper bound for the left hand side of the inequality (3.3), and thereby obtain an upper bound for $\widehat{Z}$.

Our next step requires a further Hardy-Littlewood dissection and further notation. We define the set of major arcs $\mathfrak{W}$ to be the union of the intervals

$$\mathfrak{W}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \le P^{-9/4}\},$$

with $0 \le a \le q \le P^{3/4}$ and $(a,q) = 1$. We then put $\mathfrak{w} = [0,1) \setminus \mathfrak{W}$. Recall the notation introduced in (2.12), and define also

$$S(q,a,p) = S(q,a) - p^{-1}S(q,ap^3).$$

Further, define the functions $\mathfrak{f}_p^*(\alpha)$ and $\mathfrak{g}_p^*(\alpha)$ for $\alpha \in [0,1)$ by putting

$$\mathfrak{f}_p^*(\alpha) = q^{-1}S(q,a,p)v(\alpha - a/q; P)$$

and

$$\mathfrak{g}_p^*(\alpha) = q^{-1}S(q,ap^3)v(p^3(\alpha - a/q); S),$$

when $\alpha \in \mathfrak{W}(q,a) \subseteq \mathfrak{W}$, and by setting $\mathfrak{f}_p^*(\alpha) = 0$ and $\mathfrak{g}_p^*(\alpha) = 0$ otherwise. Finally, we write

$$\mathcal{F}_1(\alpha) = \sum_{\substack{Y<p\le 2Y\\ p\equiv 2 \pmod 3}} \mathfrak{f}_p^*(\alpha)\mathfrak{g}_p^*(\alpha)\mathfrak{h}(\alpha p^3)^2.$$

The argument of [5] leading to the upper bound (3.13) of that paper reveals that

$$\int_{\mathfrak{p}\cap\mathfrak{W}} |\mathcal{F}_1(\alpha)f(\alpha)g(\alpha)|d\alpha \ll M^{-1}YS^3L^{-1/4}(\log Y)^{-1}.$$

On substituting this estimate into (3.3), we find that

$$\begin{aligned}\widehat{Z}M^{-1}YS^3(\log Y)^{-1} \ll &\int_{\mathfrak{p}\cap\mathfrak{W}} |(\mathcal{F}(\alpha) - \mathcal{F}_1(\alpha))f(\alpha)g(\alpha)K(\alpha)|d\alpha\\ &+ \int_{\mathfrak{w}} |\mathcal{F}(\alpha)f(\alpha)g(\alpha)K(\alpha)|d\alpha.\end{aligned}$$

An application of Schwarz's inequality consequently yields the bound

$$\widehat{Z}M^{-1}YS^3(\log Y)^{-1} \ll (J_1^{1/2}+J_2^{1/2})J_3^{1/2}, \tag{3.4}$$

where

$$J_1=\int_{\mathfrak{W}}|\mathcal{F}(\alpha)-\mathcal{F}_1(\alpha)|^2d\alpha,\quad J_2=\int_{\mathfrak{w}}|\mathcal{F}(\alpha)|^2d\alpha$$

and

$$J_3=\int_0^1|f(\alpha)g(\alpha)K(\alpha)|^2d\alpha.$$

But Lemma 3.2 of [5] demonstrates that

$$J_1\ll Y^2S^6P^{-19/14}\quad\text{and}\quad J_2\ll Y^2S^6P^{-19/14}.$$

Then on applying Lemma 2.1 in order to estimate J_3, and noting that our choice of parameters ensures that $P=H^2$, we deduce from (3.4) that for any positive number ε, one has

$$\widehat{Z}M^{-1}YS^3(\log Y)^{-1}\ll YS^3P^{\varepsilon-19/28}(PQ\widehat{Z}+PQ^{1/2}\widehat{Z}^{3/2})^{1/2}. \tag{3.5}$$

The proof of Theorem 1.1 may now be swiftly completed. We take $\varepsilon=\delta/4$, and recall the definitions of P, M and Q. Then we find from (3.5) that

$$\widehat{Z}\ll MQ^{1/2}P^{2\varepsilon-5/28}\widehat{Z}^{1/2}+MQ^{1/4}P^{2\varepsilon-5/28}\widehat{Z}^{3/4},$$

whence

$$\begin{aligned}\widehat{Z}&\ll M^2QP^{4\varepsilon-5/14}+M^4QP^{8\varepsilon-5/7}\\&\ll P^{17/21+\delta}+P^{11/14+2\delta}\ll N^{17/63+\delta}.\end{aligned}$$

Consequently, one has $\widehat{Z}\ll N^{\theta-\delta}=ZN^{-\delta}$, and so the conclusion of Theorem 1.1 follows as before.

4. Differencing via diminishing ranges on minor arcs. A naive application of Lemma 2.1 in pursuit of the proof of Theorem 1.2 would dictate a choice for the parameter Z lying beyond that permitted by the hypotheses of the lemma. In such circumstances, a suitable analogue of Lemma 2.1 relies on a differencing process restricted to minor arcs, and it is the object of this section to establish such a mean value estimate. Before proceeding further, we require some notation, and here we economise by recycling that employed in §2. Let P be a large positive number, take $M=P^{5/28}$, and put $Q=PM^{-1}$ and $H=PM^{-3}$. When $1\le X\le P$, we define the set of major arcs $\mathfrak{N}(X)$ to be the union of the intervals

$$\mathfrak{N}(q,a;X)=\{\alpha\in[0,1)\,:\,|q\alpha-a|\le XQ^{-3}\},$$

with $0\le a\le q\le X$ and $(a,q)=1$. We then put

$$\mathfrak{M}=\mathfrak{N}(\sqrt{P})\quad\text{and}\quad\mathfrak{m}=[0,1)\setminus\mathfrak{M}.$$

Finally, we recall the definitions of the exponential sums $f(\alpha)$, $g(\alpha)$ and $K(\alpha)$ from the preamble to Lemma 2.1.

Proposition 4.1. *Suppose that the parameters P, H, Z, and the quantity $\widehat{Z}$, satisfy*

$$H^2 \le \widehat{Z} \le Z \le P^{3/2}.$$

Then for each positive number ε, one has

$$\int_{\mathfrak{m}} |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha \ll P^\varepsilon (PQ\widehat{Z} + PH\widehat{Z}Z^{1/4} + P^{-5/4}Q^2\widehat{Z}Z). \tag{4.1}$$

We establish this proposition in several steps, and for ease of reference we summarise these steps in the shape of a sequence of lemmata. On considering the underlying diophantine equation, it follows from orthogonality that the integral

$$\int_0^1 |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha$$

is equal to the number of solutions of the diophantine equation (2.2) subject to the associated conditions. The argument initiating the proof of Lemma 2.1 shows, moreover, that whenever $\mathbf{x}$, $\mathbf{y}$, $\mathbf{n}$ is a solution of the latter equation counted by the above integral, then $|x_1 - x_2| < 3H$. Here we note that $H = Q^3P^{-2} = P^{13/28}$. Write

$$F(\alpha) = \sum_{|h|\le 3H} \sum_{\substack{P<x\le 2P \\ P<x+h\le 2P}} e(\alpha h(3x^2 + 3xh + h^2)).$$

Then on considering the underlying diophantine equations, we may infer from the above discussion that

$$\int_0^1 |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha = \int_0^1 F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha. \tag{4.2}$$

We aim now to show that the major arc contributions on the left and right hand sides of (4.2) are almost equal. From this one sees that the corresponding minor arc contributions are likewise almost equal, and since the minor arc contribution on the right hand side of (4.2) may be bounded above via conventional technology, we obtain in this way the desired upper bound (4.1). Such a procedure occurs in work of Vaughan [20] concerning Waring's problem for sixth powers.

We begin by replacing $g(\alpha)$ by its major arc approximant. First we augment the notation (2.12) by writing

$$w(\beta) = \frac{1}{3} \sum_{Q^3<m\le 8Q^3} m^{-2/3} e(\beta m).$$

We now define $g^*(\alpha)$ for $\alpha \in [0,1)$ by putting

$$g^*(\alpha) = q^{-1}S(q,a)w(\alpha - a/q),$$

when $\alpha \in \mathfrak{N}(q,a;P) \subseteq \mathfrak{N}(P)$, and by setting $g^*(\alpha) = 0$ otherwise. Finally, when ω is a complex-valued measurable function on $[0,1)$, define

$$\Xi(\omega) = \int_{\mathfrak{M}} \omega(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha - \int_{\mathfrak{M}} \omega(\alpha)|g^*(\alpha)K(\alpha)|^2 d\alpha.$$

Lemma 4.2. *One has* $\Xi(|f|^2) \ll PQ\widehat{Z}$.

Proof. From Theorem 4.1 of Vaughan [22], one finds that

$$\sup_{\alpha\in\mathfrak{M}} |g(\alpha)-g^*(\alpha)| \ll P^{1/4+\varepsilon}, \tag{4.3}$$

and hence

$$\Xi(|f|^2) \ll P^{1/2+\varepsilon}T_1 + P^{1/4+\varepsilon}T_2, \tag{4.4}$$

where

$$T_1 = \int_0^1 |f(\alpha)K(\alpha)|^2 d\alpha \quad \text{and} \quad T_2 = \int_{\mathfrak{M}} |f(\alpha)^2 g^*(\alpha)K(\alpha)^2| d\alpha. \tag{4.5}$$

The estimate $T_1 \ll P\widehat{Z}$ is immediate from the argument leading to (2.5). In order to bound T_2, we begin by observing that the methods of Chapter 4 of Vaughan [22] establish that whenever $\alpha \in \mathfrak{N}(q,a;\sqrt{P}) \subseteq \mathfrak{M}$, then one has

$$g^*(\alpha)^3 \ll Q^3(q+Q^3|q\alpha-a|)^{-1}.$$

On recalling our hypotheses on $\widehat{Z}$, it therefore follows from Lemma 2 of Brüdern [3] that

$$\int_{\mathfrak{M}} |g^*(\alpha)^3K(\alpha)^2| d\alpha \ll P^\varepsilon(P^{1/2}\widehat{Z}+\widehat{Z}^2) \ll P^\varepsilon \widehat{Z}^2. \tag{4.6}$$

Observe next that the methods of Chapter 4 of Vaughan [22] also show that whenever $\alpha \in \mathfrak{N}(q,a;\sqrt{P}) \subseteq \mathfrak{M}$, then

$$f(\alpha) \ll P(q+P^3|q\alpha-a|)^{-1/3} + q^{1/2+\varepsilon}(1+P^3|\alpha-a/q|)^{1/2}.$$

The hypotheses on M ensure that whenever $\alpha \in \mathfrak{N}(q,a;\sqrt{P}) \subseteq \mathfrak{M}$, then

$$q^\varepsilon(q+P^3|q\alpha-a|) \ll P^{1/2+\varepsilon}M^3 = P^{29/28+\varepsilon} \ll P^{6/5-2\varepsilon}.$$

Thus, under the same conditions on α, one has

$$f(\alpha)^3 \ll P^3(q+P^3|q\alpha-a|)^{-1},$$

whence Lemma 2 of Brüdern [3] yields

$$\int_{\mathfrak{M}} |f(\alpha)^3K(\alpha)^2| d\alpha \ll P^\varepsilon(P^{1/2}\widehat{Z}+\widehat{Z}^2) \ll P^\varepsilon \widehat{Z}^2. \tag{4.7}$$

Finally, an application of Hölder's inequality leads from (4.5), via (4.6) and (4.7), to the upper bound

$$T_2 \le \left(\int_{\mathfrak{M}} |f(\alpha)^3K(\alpha)^2| d\alpha\right)^{2/3} \left(\int_{\mathfrak{M}} |g^*(\alpha)^3K(\alpha)^2| d\alpha\right)^{1/3} \ll P^\varepsilon \widehat{Z}^2. \tag{4.8}$$

On substituting (4.8) along with our earlier bound for T_1 into (4.4), and noting that our hypotheses ensure that $Q \gg P^{3/4+\varepsilon}$ and $\widehat{Z} \ll P^{3/2}$, we conclude that

$$\Xi(|f|^2) \ll P^{3/2+\varepsilon}\widehat{Z} + P^{1/4+\varepsilon}\widehat{Z}^2 \ll PQ\widehat{Z}.$$

This completes the proof of the lemma.

We next establish a conclusion similar to that of Lemma 4.2 in which $|f|^2$ is replaced by F.

Lemma 4.3. *One has* $\Xi(F) \ll PQ\widehat{Z}$.

Proof. In view of (4.3), one has

$$\Xi(F) \ll P^{1/2+\varepsilon}T_3 + P^{1/4+\varepsilon}T_4, \tag{4.9}$$

where

$$T_3 = \int_{\mathfrak{M}} |F(\alpha)K(\alpha)^2|d\alpha \quad \text{and} \quad T_4 = \int_{\mathfrak{M}} |F(\alpha)g^*(\alpha)K(\alpha)^2|d\alpha. \tag{4.10}$$

It is convenient for future reference to define the function $\Upsilon(\alpha)$ on $[0,1)$ by taking

$$\Upsilon(\alpha) = (q + Q^3|q\alpha - a|)^{-1},$$

when $\alpha \in \mathfrak{N}(q,a;H^2) \subseteq \mathfrak{N}(H^2)$, and by setting $\Upsilon(\alpha) = 0$ otherwise. On isolating the term in $F(\alpha)$ corresponding to the diagonal contribution with $h = 0$, it follows from the argument of the proof of Lemma 3.1 of Vaughan [21] (just as in the derivation of (2.6) above) that

$$\begin{aligned} F(\alpha) &\ll P^{1+\varepsilon} + P^{1/2+\varepsilon}H + P^{1+\varepsilon}H\Upsilon(\alpha)^{1/2} \\ &\ll P^{1+\varepsilon} + P^{1+\varepsilon}H\Upsilon(\alpha)^{1/2}, \end{aligned} \tag{4.11}$$

uniformly in $\alpha \in [0,1)$. When $\alpha \in \mathfrak{M}$, therefore, one has

$$F(\alpha) \ll P^{5/4+\varepsilon}H\Upsilon(\alpha),$$

whence we deduce from Lemma 2 of Brüdern [3] that

$$T_3 \ll P^{5/4+\varepsilon}H \int_{\mathfrak{M}} \Upsilon(\alpha)|K(\alpha)|^2 d\alpha \ll P^{5/4+\varepsilon}HQ^{\varepsilon-3}(P^{1/2}\widehat{Z} + \widehat{Z}^2). \tag{4.12}$$

We have already remarked that when $\alpha \in \mathfrak{M}$, one has $g^*(\alpha) \ll Q\Upsilon(\alpha)^{1/3}$, and thus it follows from (4.11) that whenever $\alpha \in \mathfrak{M}$,

$$F(\alpha)g^*(\alpha) \ll P^{1+\varepsilon}HQ\Upsilon(\alpha)^{5/6} \ll HP^{13/12+\varepsilon}Q\Upsilon(\alpha).$$

Consequently, again by Lemma 2 of Brüdern [3], we find from (4.10) that

$$T_4 \ll HP^{13/12+\varepsilon}Q \int_{\mathfrak{M}} \Upsilon(\alpha)|K(\alpha)|^2 d\alpha \ll HP^{13/12+\varepsilon}Q^{\varepsilon-2}(P^{1/2}\widehat{Z} + \widehat{Z}^2). \tag{4.13}$$

Finally, on substituting (4.12) and (4.13) into (4.9), and recalling our hypotheses concerning P, Q and $\widehat{Z}$, we obtain the upper bound

$$\begin{aligned} \Xi(F) &\ll P^{\varepsilon}(P^{7/4}HQ^{-3} + HP^{4/3}Q^{-2})\widehat{Z}^2 \\ &\ll P^{\varepsilon}(P^{-1/4} + P^{1/3}M^{-1})\widehat{Z}^2 \ll PQ\widehat{Z}. \end{aligned}$$

This completes the proof of the lemma.

Our next objective is the completion of the singular integrals implicitly associated with F and $|f|^2$. The dependence on q of the width of our major arcs generates several difficulties in this process. For the sake of concision, write $\Phi(\alpha)=F(\alpha)|K(\alpha)|^2$. Also, define the multiplicative function $\kappa(q)$ on prime powers π^l ($l\in\mathbb{N}$) by means of the relations

$$\kappa(\pi^{3l})=\pi^{-l},\quad \kappa(\pi^{3l+1})=2\pi^{-l-1/2}\quad\text{and}\quad \kappa(\pi^{3l+2})=\pi^{-l-1}. \tag{4.14}$$

We note for future reference that it follows from Lemmata 4.3–4.5 of Vaughan [22] that whenever $q\in\mathbb{N}$ and $a\in\mathbb{Z}$ satisfy $(a,q)=1$, then one has $q^{-1}S(q,a)\ll\kappa(q)$. Finally, when Θ is a complex-valued measurable function on $[0,1)$, we write

$$\Omega(\Theta)=\sum_{1\le q\le\sqrt{P}}\sum_{\substack{a=1\\(a,q)=1}}^{q}q^{-2}|S(q,a)|^2\int_{-1/2}^{1/2}|w(\beta)|^2\Theta(\beta+a/q)d\beta. \tag{4.15}$$

Lemma 4.4. *One has*

$$\int_{\mathfrak{M}}F(\alpha)|g^*(\alpha)K(\alpha)|^2d\alpha=\Omega(\Phi)+O(HP^{3/4+\varepsilon}Q^{-1}\widehat{Z}Z).$$

Proof. Observe first that whenever $1\le q\le\sqrt{P}$ and

$$\sqrt{P}/(qQ^3)<|\beta|\le H^2/(qQ^3), \tag{4.16}$$

then the estimate (4.11) shows that

$$F(\beta+a/q)\ll P^{1+\varepsilon}+P^{3/4+\varepsilon}H\ll P^{3/4+\varepsilon}H.$$

It therefore follows from the upper bound $w(\beta)\ll Q(1+Q^3|\beta|)^{-1}$, provided by Lemma 6.2 of Vaughan [22], that the estimate

$$|w(\beta)^2\Phi(\beta+a/q)|\ll HP^{3/4+\varepsilon}Q^2(1+Q^3|\beta|)^{-1}|K(\beta+a/q)|^2 \tag{4.17}$$

holds throughout the range (4.16). Our hypotheses ensure that $H^2P^{-1/2}>P^{1/4}$, and so whenever $q\le\sqrt{P}$ and $|\beta|>H^2/(qQ^3)$, it follows that

$$|\beta|>H^2P^{-1/2}Q^{-3}>P^{1/4}Q^{-3}.$$

Our earlier estimate for $w(\beta)$ therefore leads to the bound

$$|w(\beta)|^2\ll Q^2(1+Q^3|\beta|)^{-2}\ll Q^2P^{-1/4}(1+Q^3|\beta|)^{-1},$$

and so the upper bound (4.17) now follows, from a trivial estimate for $F(\alpha)$, also in the range $|\beta| > H^2/(qQ^3)$. We thus conclude from (4.15) that

$$\int_{\mathfrak{M}} F(\alpha)|g^*(\alpha)K(\alpha)|^2 d\alpha - \Omega(\Phi) \ll HP^{3/4+\varepsilon}Q^2\Lambda(\sqrt{P}), \tag{4.18}$$

where we write

$$\Lambda(U) = \sum_{1\le q\le U} \kappa(q)^2 \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{-1/2}^{1/2} (1+Q^3|\beta|)^{-1}|K(\beta+a/q)|^2 d\beta. \tag{4.19}$$

From (4.19), we have

$$\Lambda(U) = \sum_{1\le q\le U} \kappa(q)^2 \sum_{n,m\in\mathcal{Z}} c_q(n-m) \int_{-1/2}^{1/2} \frac{e(\beta(n-m))}{1+Q^3|\beta|} d\beta,$$

where

$$c_q(h) = \sum_{\substack{a=1\\(a,q)=1}}^{q} e(ah/q)$$

is Ramanujan's sum. The standard estimate $|c_q(h)| \le (q,h)$ implies that

$$\Lambda(U) \ll Q^{\varepsilon-3}(T_5+T_6), \tag{4.20}$$

where

$$T_5 = \widehat{Z} \sum_{1\le q\le U} q\kappa(q)^2 \quad \text{and} \quad T_6 = \sum_{1\le q\le U} \kappa(q)^2 \sum_{\substack{n,m\in\mathcal{Z}\\ n\ne m}} (q,n-m). \tag{4.21}$$

But whenever $n,m\in\mathcal{Z}$ satisfy $n\ne m$, one has $1\le |n-m|\le Z$. Thus we find that

$$\sum_{\substack{n,m\in\mathcal{Z}\\ n\ne m}} (q,n-m) \ll \widehat{Z} \sum_{1\le l\le Z} (q,l) \ll \widehat{Z}Zq^{\varepsilon}. \tag{4.22}$$

The relations (4.14) imply, moreover, that

$$\sum_{1\le q\le U} \kappa(q)^2 \ll \prod_{p\le U} (1+4p^{-1}) \ll (\log U)^4,$$

and so we conclude from (4.20)–(4.22) that

$$\Lambda(U) \ll U^{\varepsilon}Q^{\varepsilon-3}(U\widehat{Z}+\widehat{Z}Z). \tag{4.23}$$

In view of (4.18), therefore, we have the upper bound

$$\begin{aligned}\int_{\mathfrak{M}} F(\alpha)|g^*(\alpha)K(\alpha)|^2 d\alpha - \Omega(\Phi) &\ll HP^{3/4+\varepsilon}Q^{-1}(P^{1/2}\widehat{Z}+\widehat{Z}Z)\\ &\ll HP^{3/4+\varepsilon}Q^{-1}\widehat{Z}Z,\end{aligned}$$

and this completes the proof of the lemma.

We also require an analogue of Lemma 4.4 in which $F(\alpha)$ is replaced by $|f(\alpha)|^2$. We now write $\Psi(\alpha) = |f(\alpha)K(\alpha)|^2$, and recall the notation defined in (4.15).

Lemma 4.5. *One has*

$$\int_{\mathfrak{M}} |f(\alpha)g^*(\alpha)K(\alpha)|^2 d\alpha = \Omega(\Psi) + O(HP^{3/4+\varepsilon}Q^{-1}\widehat{Z}Z).$$

Proof. We begin by observing that the methods of Chapters 4 and 6 of Vaughan [22] demonstrate that whenever $\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, and $a \in \mathbb{Z}$, $q \in \mathbb{N}$ satisfy $(a, q) = 1$, then

$$|f(\beta + a/q)|^2 \ll q^{-2/3}P^2(1 + P^3|\beta|)^{-2} + q^{1+\varepsilon}(1 + P^3|\beta|).$$

Our earlier estimate for $w(\beta)$ therefore reveals that

$$|f(\beta + a/q)w(\beta)|^2 \ll \frac{P^2Q^2}{1 + Q^3|\beta|}(T_7 + T_8), \tag{4.24}$$

where

$$T_7 = q^{-2/3}(1 + P^3|\beta|)^{-2}(1 + Q^3|\beta|)^{-1}$$

and

$$T_8 = q^{1+\varepsilon}P^{-2}(1 + P^3|\beta|)(1 + Q^3|\beta|)^{-1}.$$

On considering separately the cases in which $|\beta| \geq Q^{-3}$, and $|\beta| < Q^{-3}$, respectively, it is apparent that for all β one has

$$(1 + P^3|\beta|)(1 + Q^3|\beta|)^{-1} \ll P^3Q^{-3} = M^3.$$

Consequently, whenever $q \leq \sqrt{P}$, one has

$$T_8 \ll P^{\varepsilon-3/2}M^3 = P^{\varepsilon-27/28} \ll P^{-11/14},$$

and if in addition β satisfies $|\beta| \geq \sqrt{P}/(qQ^3)$, then

$$T_7 \ll q^{-2/3}M^3(\sqrt{P}M^3/q)^{-3} \ll (\sqrt{P})^{-2/3}M^{-6} \ll P^{-11/14}.$$

Thus we conclude from (4.24) that whenever $q \leq \sqrt{P}$ and $|\beta| \geq \sqrt{P}/(qQ^3)$, then

$$|f(\beta + a/q)w(\beta)|^2 \ll Q^2P^{17/14}(1 + Q^3|\beta|)^{-1}. \tag{4.25}$$

On recalling the definition of $\Psi(\alpha)$, it follows from (4.25) that

$$\int_{\mathfrak{M}} |f(\alpha)g^*(\alpha)K(\alpha)|^2 d\alpha - \Omega(\Psi) \ll Q^2P^{17/14}\Lambda(\sqrt{P}),$$

where Λ is defined as in (4.19). The estimate (4.23) consequently leads to the bound

$$\begin{aligned}\int_{\mathfrak{M}} |f(\alpha)g^*(\alpha)K(\alpha)|^2 d\alpha - \Omega(\Psi) &\ll (Q^2P^{17/14})Q^{\varepsilon-3}(P^{1/2}\widehat{Z} + \widehat{Z}Z)\\ &\ll Q^{\varepsilon-1}P^{17/14}\widehat{Z}Z.\end{aligned}$$

The conclusion of the lemma follows on recalling that $Q = P^{23/28}$ and $H = P^{13/28}$.

The main terms in the expansions established in Lemmata 4.4 and 4.5 are in fact equal, as we now demonstrate.

Lemma 4.6. *One has* $\Omega(\Phi) = \Omega(\Psi)$.

Proof. Write

$$J(l) = \int_{-1/2}^{1/2} |w(\beta)|^2 e(\beta l) d\beta,$$

and note that the definition of $w(\beta)$ ensures that $J(l) = 0$ for $|l| > 8Q^3$. It is convenient also to write

$$\chi(l) = e(la/q)J(l).$$

Then it follows from the definition of Ψ that

$$\int_{-1/2}^{1/2} |w(\beta)|^2 \Psi(\beta + a/q) d\beta = \sum_{P<x,y\leq 2P} \sum_{m,n\in\mathcal{Z}} \chi(x^3 - y^3 + n - m). \tag{4.26}$$

Thus, just as in the argument leading to (4.2) above, the condition that $J(l) = 0$ for $|l| > 8Q^3$ implies that the only values of x and y contributing to the sum in (4.26) are those with $|x - y| < 3H$. Following a change of variable, we deduce that

$$\begin{aligned} &\int_{-1/2}^{1/2} |w(\beta)|^2 \Psi(\beta + a/q) d\beta \\ &\quad = \sum_{|h|\leq 3H} \sum_{\substack{P<x\leq 2P \\ P<x+h\leq 2P}} \sum_{m,n\in\mathcal{Z}} \chi(h(3x^2 + 3xh + h^2) + n - m) \\ &\quad = \int_{-1/2}^{1/2} |w(\beta)|^2 \Phi(\beta + a/q) d\beta. \end{aligned}$$

The conclusion of the lemma is now immediate from (4.15).

Collecting together the conclusions of Lemmata 4.4–4.6, we see that

$$\int_{\mathfrak{M}} |f(\alpha)g^*(\alpha)K(\alpha)|^2 d\alpha - \int_{\mathfrak{M}} F(\alpha)|g^*(\alpha)K(\alpha)|^2 d\alpha \ll HP^{3/4+\varepsilon}Q^{-1}\widehat{Z}Z,$$

so that in view of Lemmata 4.2 and 4.3, we have

$$\int_{\mathfrak{M}} |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha - \int_{\mathfrak{M}} F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha \ll PQ\widehat{Z} + HP^{3/4+\varepsilon}Q^{-1}\widehat{Z}Z.$$

Finally, making use now of the relation (4.2), we conclude thus far that

$$\int_{\mathfrak{m}} |f(\alpha)g(\alpha)K(\alpha)|^2 d\alpha - \int_{\mathfrak{m}} F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha \ll PQ\widehat{Z} + HP^{3/4+\varepsilon}Q^{-1}\widehat{Z}Z. \tag{4.27}$$

Our objective now is to estimate the second integral on the left hand side of (4.27), and it transpires that a satisfactory bound here delivers the sought after conclusion of Proposition 4.1.

Lemma 4.7. *One has*

$$\int_{\mathfrak{m}} F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha \ll P^{\varepsilon}(PQ\widehat{Z} + PH\widehat{Z}Z^{1/4} + P^{-5/4}Q^2\widehat{Z}Z).$$

Proof. Write $\mathfrak{N} = \mathfrak{N}(H^2)$ and $\mathfrak{n} = [0,1) \setminus \mathfrak{N}$. Then it follows from (4.11) that

$$\sup_{\alpha \in \mathfrak{n}} |F(\alpha)| \ll P^{1+\varepsilon}.$$

The argument leading to (2.5) above, moreover, shows on this occasion that

$$\int_0^1 |g(\alpha)K(\alpha)|^2 d\alpha \ll Q\widehat{Z}. \tag{4.28}$$

Thus we find that

$$\begin{aligned}\int_{\mathfrak{n}} F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha &\ll \Big(\sup_{\alpha \in \mathfrak{n}} |F(\alpha)|\Big) \int_0^1 |g(\alpha)K(\alpha)|^2 d\alpha \\ &\ll P^{1+\varepsilon} Q\widehat{Z}. \end{aligned} \tag{4.29}$$

Observe next that (4.11) provides the upper bound

$$F(\alpha) \ll P^{1+\varepsilon} H \Upsilon(\alpha)^{1/2}, \tag{4.30}$$

valid uniformly for $\alpha \in \mathfrak{N}$. We deduce from (4.30) that

$$\int_{\mathfrak{N}\cap\mathfrak{m}} F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha \ll P^{1+\varepsilon} H T_9, \tag{4.31}$$

where

$$T_9 = \int_{\mathfrak{N}\cap\mathfrak{m}} \Upsilon(\alpha)^{1/2} |g(\alpha)K(\alpha)|^2 d\alpha. \tag{4.32}$$

But Theorem 4.1 of Vaughan [22] demonstrates that for $\alpha \in \mathfrak{N}$,

$$|g(\alpha)|^{1/2} \ll |g^*(\alpha)|^{1/2} + P^{\varepsilon} \Upsilon(\alpha)^{-1/4}. \tag{4.33}$$

On substituting (4.33) into (4.32), we find that

$$T_9 \ll T_{10} + P^{\varepsilon} T_{11}, \tag{4.34}$$

where

$$T_{10} = \int_{\mathfrak{N}\cap\mathfrak{m}} \Upsilon(\alpha)^{1/2} |g^*(\alpha) g(\alpha)^3|^{1/2} |K(\alpha)|^2 d\alpha \tag{4.35}$$

and

$$T_{11} = \int_{\mathfrak{N}\cap\mathfrak{m}} \Upsilon(\alpha)^{1/4} |g(\alpha)|^{3/2} |K(\alpha)|^2 d\alpha. \tag{4.36}$$

But an application of Hölder's inequality to (4.35) reveals that

$$T_{10} \leq T_9^{3/4} T_{12}^{1/4}, \tag{4.37}$$

where

$$T_{12} = \int_{\mathfrak{N}\cap\mathfrak{m}} \Upsilon(\alpha)^{1/2} |g^*(\alpha)K(\alpha)|^2 d\alpha. \tag{4.38}$$

We thus conclude from (4.34) and (4.37) that

$$T_9 \ll P^\varepsilon (T_{11} + T_{12}). \tag{4.39}$$

From Lemma 2 of Brüdern [3] one has

$$\int_{\mathfrak{N}} \Upsilon(\alpha)|K(\alpha)|^2 d\alpha \ll Q^{\varepsilon-3}(H^2\widehat{Z} + \widehat{Z}^2) \ll Q^{\varepsilon-3}\widehat{Z}^2.$$

By applying Hölder's inequality to (4.36), and recalling (4.28), we therefore find that

$$\begin{aligned} T_{11} &\leq \Big(\int_{\mathfrak{N}} \Upsilon(\alpha)|K(\alpha)|^2 d\alpha\Big)^{1/4} \Big(\int_0^1 |g(\alpha)K(\alpha)|^2 d\alpha\Big)^{3/4} \\ &\ll (Q^{\varepsilon-3}\widehat{Z}^2)^{1/4}(Q\widehat{Z})^{3/4} \ll P^\varepsilon \widehat{Z}^{5/4}. \end{aligned} \tag{4.40}$$

Meanwhile, on noting that $\Upsilon(\alpha) \ll P^{-1/2}$ for $\alpha \in \mathfrak{N}\cap\mathfrak{m}$, we find from (4.38) that

$$T_{12} \ll P^{-1/4} \int_{\mathfrak{N}\cap\mathfrak{m}} |g^*(\alpha)K(\alpha)|^2 d\alpha. \tag{4.41}$$

But on recalling (4.19) together with the upper bound (4.23), one has

$$\int_{\mathfrak{N}} |g^*(\alpha)K(\alpha)|^2 d\alpha \ll Q^2\Lambda(H^2) \ll Q^{\varepsilon-1}(H^2\widehat{Z} + \widehat{Z}Z) \ll Q^{\varepsilon-1}\widehat{Z}Z.$$

We therefore conclude from (4.41) that

$$T_{12} \ll P^{\varepsilon-1/4}Q^{-1}\widehat{Z}Z,$$

so that by collecting together (4.31), (4.39) and (4.40) together with this most recent upper bound, we infer that

$$\int_{\mathfrak{N}\cap\mathfrak{m}} F(\alpha)|g(\alpha)K(\alpha)|^2 d\alpha \ll P^{1+\varepsilon}H(\widehat{Z}Z^{1/4} + P^{-1/4}Q^{-1}\widehat{Z}Z).$$

The conclusion of the lemma follows from this bound together with (4.29).

The conclusion of Proposition 4.1 is now immediate on substituting the estimate provided by Lemma 4.7 into (4.27).

5. Sums of five cubes in short intervals. We are now equipped to prove Theorem 1.2. Let N be a large natural number, write $P = \frac{1}{2}N^{1/3}$, and define M, Q and H as in §4. Let δ be a sufficiently small positive number, and put $Z = P^{10/7}$. Define $\mathcal{Z} = \mathcal{Z}(N, Z)$ to be the set of integers n with $N < n \le N + Z$ that cannot be written as the sum of 5 cubes of natural numbers. We aim to establish that $\widehat{Z} \ll P^{10/7-\delta}$. We may assume without loss, therefore, that $\widehat{Z} \gg P^{10/7-\delta}$. As in the discussion at the beginning of §3, the conclusion of Theorem 1.2 follows in general from this restricted conclusion for the latter choice of Z. We continue to make use of the exponential sums $f(\alpha)$ and $g(\alpha)$ from §4, but now introduce the smooth Weyl sum

$$t(\alpha) = \sum_{x \in \mathcal{A}(P, P^\eta)} e(\alpha x^3),$$

where, as usual, we suppose that η is a sufficiently small positive number. Write

$$\rho(n) = \int_0^1 f(\alpha)^2 g(\alpha) t(\alpha)^2 e(-n\alpha) d\alpha. \tag{5.1}$$

Then whenever $n \in \mathcal{Z}$, one has $\rho(n) = 0$. Defining the exponential sum $K(\alpha)$ as in §2 (and also, implicitly, as in §4), it follows from (5.1) that

$$\int_0^1 f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha) d\alpha = 0. \tag{5.2}$$

We begin with a major arc estimate.

Lemma 5.1. *One has*

$$\int_{\mathfrak{M}} f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha) d\alpha \gg PQ\widehat{Z}.$$

Proof. Write $L = (\log N)^{1/100}$, and define the narrow set of major arcs $\mathfrak{P}$ as in the preamble to Lemma 3.1. Also, when $1 \le X \le P^{3/2}$, let $\mathfrak{W}(X)$ denote the union of the intervals

$$\mathfrak{W}(q, a; X) = \{\alpha \in [0, 1) : |q\alpha - a| \le XP^{-3}\},$$

with $0 \le a \le q \le X$ and $(a, q) = 1$. We then put $\mathfrak{K}(X) = \mathfrak{W}(2X) \setminus \mathfrak{W}(X)$.

The methods of Chapter 4 of Vaughan [22] show that whenever $\alpha \in \mathfrak{N}(q, a; \sqrt{P}) \subseteq \mathfrak{M}$, then

$$g(\alpha) \ll Q(q + Q^3|q\alpha - a|)^{-1/3} + q^{1/2+\varepsilon} \ll Qq^{-1/3},$$

and

$$\begin{aligned} f(\alpha) &\ll \kappa(q)P(1 + P^3|\alpha - a/q|)^{-1} + q^\varepsilon(q + P^3|q\alpha - a|)^{1/2} \\ &\ll \kappa(q)P(1 + P^3|\alpha - a/q|)^{-1} + P^{1/4+\varepsilon}M^{3/2}. \end{aligned}$$

But whenever $\alpha \in \mathfrak{N}(q,a;\sqrt{P}) \subseteq \mathfrak{M}$ and $\alpha \in \mathfrak{K}(X)$, one necessarily has

$$q + P^3|q\alpha - a| > X,$$

and thus it follows, under the same conditions on α, that

$$\begin{aligned} f(\alpha)^2 g(\alpha) &\ll P^{1/2+\varepsilon} M^3 |g(\alpha)| \\ &\qquad + \kappa(q)^2 P^2 Q (1 + P^3|\alpha - a/q|)^{-5/3} (q + P^3|q\alpha - a|)^{-1/3} \\ &\ll P^{1/2+\varepsilon} M^3 |g(\alpha)| + \kappa(q)^2 P^2 Q X^{-1/3} (1 + P^3|\alpha - a/q|)^{-5/3}. \end{aligned}$$

We therefore conclude that

$$\int_{\mathfrak{M}\cap\mathfrak{K}(X)} f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha)\,d\alpha \ll T_{13} + T_{14}, \tag{5.3}$$

where

$$T_{13} = P^{1/2+\varepsilon} M^3 \int_0^1 |g(\alpha) t(\alpha)^2 K(\alpha)|\,d\alpha, \tag{5.4}$$

$$T_{14} = P^2 Q X^{-1/3} \widehat{Z} \sum_{1\le q\le 2X} \kappa(q)^2 \tau(q), \tag{5.5}$$

and

$$\tau(q) = \int_{-\infty}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} |t(\beta + a/q)|^2 (1 + P^3|\beta|)^{-5/3}\,d\beta.$$

By Schwarz's inequality, one has

$$\int_0^1 |g(\alpha) t(\alpha)^2 K(\alpha)|\,d\alpha \le \Big(\int_0^1 |g(\alpha) K(\alpha)|^2 d\alpha \Big)^{1/2} \Big(\int_0^1 |t(\alpha)|^4 d\alpha \Big)^{1/2}. \tag{5.6}$$

Here, the first integral on the right hand side of (5.6) may be estimated via (4.28), and the second by means of Hua's lemma (see, for example, Lemma 2.5 of Vaughan [22]), and hence we deduce from (5.4) that

$$T_{13} \ll P^{1/2+\varepsilon} M^3 (Q\widehat{Z})^{1/2} (P^{2+\varepsilon})^{1/2} \ll P^{3/2+2\varepsilon} Q^{1/2} M^3 \widehat{Z}^{1/2}. \tag{5.7}$$

The expression (5.5), on the other hand, may be estimated via the argument of the proof of Lemma 3.4 of Brüdern, Kawada and Wooley [5] (see also the proof of Lemma 3.3 of the latter paper). In this way, one obtains

$$T_{14} \ll (P^2 Q X^{-1/3} \widehat{Z})(X^\varepsilon P^{-1}) \ll PQ\widehat{Z} X^{-1/4}. \tag{5.8}$$

On substituting (5.7) and (5.8) into (5.3), and summing over the values $X = 2^i L$ with $X \le \sqrt{P}$ and $i \ge 0$, we may conclude thus far that

$$\int_{\mathfrak{M}\setminus\mathfrak{P}} f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha)\,d\alpha \ll P^{3/2+\varepsilon} Q^{1/2} M^3 \widehat{Z}^{1/2} + PQ\widehat{Z} L^{-1/4}. \tag{5.9}$$

The set of major arcs $\mathfrak{P}$ are sufficiently few and narrow that arguments nowadays considered routine (see §5 of Vaughan [21]) suffice to establish that for each integer n with $N < n \le N + Z$, one has

$$\int_{\mathfrak{P}} f(\alpha)^2 g(\alpha) t(\alpha)^2 e(-n\alpha)\, d\alpha \gg PQ.$$

It follows that

$$\int_{\mathfrak{P}} f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha)\, d\alpha \gg \sum_{n \in \mathcal{Z}} PQ = PQ\widehat{Z}.$$

On recalling (5.9) and noting our hypotheses on H, P, Z and $\widehat{Z}$, a modest computation confirms the desired lower bound.

We now return to the identity (5.2). Since $[0,1)$ is the disjoint union of $\mathfrak{M}$ and $\mathfrak{m}$, it follows from Lemma 5.1 that

$$\int_{\mathfrak{m}} f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha)\, d\alpha = -\int_{\mathfrak{M}} f(\alpha)^2 g(\alpha) t(\alpha)^2 K(-\alpha)\, d\alpha,$$

whence

$$\int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha) t(\alpha)^2 K(\alpha)|\, d\alpha \gg PQ\widehat{Z}. \tag{5.10}$$

But by Schwarz's inequality, one has

$$\int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha) t(\alpha)^2 K(\alpha)|\, d\alpha \le \Big(\int_{\mathfrak{m}} |f(\alpha) g(\alpha) K(\alpha)|^2\, d\alpha \Big)^{1/2} \Big(\int_0^1 |f(\alpha)^2 t(\alpha)^4|\, d\alpha \Big)^{1/2}. \tag{5.11}$$

On considering the underlying diophantine equation, it follows from Theorem 1.2 of Wooley [23] that

$$\int_0^1 |f(\alpha)^2 t(\alpha)^4|\, d\alpha \ll P^{13/4 - 9\delta},$$

provided that we take $\delta < 10^{-5}$. Then on substituting this estimate together with the conclusion of Proposition 4.1 into (5.11), we find that

$$\int_{\mathfrak{m}} |f(\alpha)^2 g(\alpha) t(\alpha)^2 K(\alpha)|\, d\alpha \ll (PQ\widehat{Z} + PH\widehat{Z}Z^{1/4} + P^{-5/4}Q^2\widehat{Z}Z)^{1/2} (P^{13/8 - 4\delta}).$$

Consequently, the lower bound (5.10) implies that

$$PQ\widehat{Z} \ll P^{17/8 - 4\delta} Q^{1/2} \widehat{Z}^{1/2} + P^{17/8 - 4\delta} H^{1/2} \widehat{Z}^{1/2} Z^{1/8} + P^{1 - 4\delta} Q (\widehat{Z}Z)^{1/2}.$$

On recalling the definitions of the relevant parameters, we thus arrive first at the estimate

$$\widehat{Z} \ll \widehat{Z}^{1/2}(P^{5/8-4\delta}M^{1/2} + P^{5/8-4\delta}M^{-1/2}(P^{10/7})^{1/8} + P^{5/7-4\delta}),$$

and then

$$\widehat{Z} \ll (P^{5/7-4\delta})^2 + P^{10/7-8\delta} \ll P^{10/7-8\delta}.$$

The relation $P \asymp N^{1/3}$ consequently leads to the upper bound

$$\widehat{Z} \ll N^{10/21-\delta} \ll ZN^{-\delta},$$

and the conclusion of Theorem 1.2 follows by summing over the blocks discussed at the start of section 3.

6. The asymptotic formula for sums of cubes. We turn our attention in this section to the proof of Theorem 1.3. It transpires that the argument here is far less involved than that of §§2 and 3, and can be modelled closely on the analysis described in §2 of Brüdern, Kawada and Wooley [7].

Let t be an integer with $1 \le t \le 3$, and write $s = 4 + t$. Also, let $\psi(\tau) = \psi_s(\tau)$ be a function of the type described in the statement of Theorem 1.3. We consider large positive numbers M and N with $M \le N$, and define $\mathcal{Z}_s = \mathcal{Z}_s(N, M)$ to be the set of integers n with $N < n \le N + M$ for which the lower bound (1.6) holds. It is convenient to abbreviate $\mathrm{card}(\mathcal{Z}_s(N, M))$ simply to $Z_s = Z_s(N, M)$. As in the discussion of the first paragraph of §3, the conclusion of Theorem 1.3 follows on demonstrating that whenever $M \le \frac{1}{2}N^{2/3}$, then one has

$$Z_s \ll N^{(9-s)/6}(\log N)^{\varepsilon-(11-s)/2}\psi_s(N)^2 \quad (5 \le s \le 7). \tag{6.1}$$

For if $M > \frac{1}{2}N^{2/3}$, then we may subdivide the interval $(N, N + M]$ into at most $[2MN^{-2/3}] + 1$ subintervals $(N_0, N_0 + \frac{1}{2}N_0^{2/3}]$ of length $\frac{1}{2}N_0^{2/3}$, on each of which we may apply the bound (6.1). We thus obtain the estimate

$$Z_s(N, M) \ll (MN^{-2/3} + 1)N^{(9-s)/6}(\log N)^{\varepsilon-(11-s)/2}\psi_s(N)^2,$$

and the conclusion of Theorem 1.3 follows on noting that

$$Z_s(N, M) = \widetilde{E}_s(N + M; \psi_s) - \widetilde{E}_s(N; \psi_s).$$

We take $P = \frac{1}{2}N^{1/3}$, and define the exponential sums

$$h(\alpha) = \sum_{1 \le x \le 3P} e(\alpha x^3) \quad \text{and} \quad h_1(\alpha) = \sum_{P < y \le 3P} e(\alpha y^3).$$

In view of the discussion of the previous paragraph, we may suppose in what follows that $1 \le M \le \frac{1}{2}N^{2/3} = 2P^2$. Consider an integer n with $N < n \le N + M$, and suppose that $x_1, \dots, x_s$ are natural numbers satisfying the equation

$$n = x_1^3 + \cdots + x_s^3.$$

It is apparent that one necessarily has

$$\max_{1\le i\le s} x_i \ge (n/s)^{1/3} > (N/8)^{1/3} = P, \tag{6.2}$$

as well as

$$\max_{1\le i\le s} x_i \le n^{1/3} \le (2N)^{1/3} < 3P. \tag{6.3}$$

By orthogonality, it follows from (6.2) that

$$\int_0^1 (h(\alpha)-h_1(\alpha))^s e(-n\alpha)d\alpha = 0,$$

and from (6.3) we see that

$$R_s(n) = \int_0^1 h(\alpha)^s e(-n\alpha)d\alpha.$$

On substituting the former into the latter, it follows that

$$\begin{aligned} R_s(n) &= \int_0^1 \left(h(\alpha)^s - (h(\alpha)-h_1(\alpha))^s\right) e(-n\alpha)d\alpha \\ &= \sum_{j=1}^s (-1)^{j+1}\binom{s}{j}\mathcal{R}_{s,j}([0,1)), \end{aligned} \tag{6.4}$$

where we write

$$\mathcal{R}_{s,j}(\mathfrak{B}) = \int_{\mathfrak{B}} h_1(\alpha)^j h(\alpha)^{s-j} e(-n\alpha)d\alpha. \tag{6.5}$$

Let $\mathfrak{M}$ denote the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha\in[0,1) : |q\alpha - a| \le P(6N)^{-1}\},$$

with $0 \le a \le q \le P/6$ and $(a,q)=1$. Also, recall the definition (1.5) of the singular series $\mathfrak{S}_s(n)$. Then it follows from the methods underlying the proof of Theorem 4.4 of Vaughan [22] that there is a positive number ν such that whenever $N < n \le N+M$, one has

$$\sum_{j=1}^s (-1)^{j+1}\binom{s}{j}\mathcal{R}_{s,j}(\mathfrak{M}) = \frac{\Gamma(4/3)^s}{\Gamma(s/3)}\mathfrak{S}_s(n)n^{s/3-1} + O(n^{s/3-1-\nu}). \tag{6.6}$$

Since this observation is not quite transparent, we offer some additional explanation. On following the argument of the proof of Theorem 4.4 of Vaughan [22], one finds that for $1 \le j \le s$ one has

$$\mathcal{R}_{s,j}(\mathfrak{M}) = J_{s,j}(n)\mathfrak{S}_s(n) + O(n^{s/3-1-\nu}), \tag{6.7}$$

where

$$J_{s,j}(n)=\int_{-\infty}^{\infty}\int_{\mathfrak{B}_{s,j}} e(\beta(\gamma_1^3+\cdots+\gamma_s^3-n))d\boldsymbol{\gamma}d\beta,$$

and

$$\mathfrak{B}_{s,j}=[P,3P]^j\times[0,3P]^{s-j}.$$

It follows that

$$\sum_{j=1}^{s}(-1)^{j+1}\binom{s}{j}J_{s,j}(n)=J(3P)-J(P), \tag{6.8}$$

where we write

$$J(Q)=\int_{-\infty}^{\infty}\int_{[0,Q]^s} e(\beta(\gamma_1^3+\cdots+\gamma_s^3-n))d\boldsymbol{\gamma}d\beta.$$

But an application of Fourier's integral formula, just as in the classical treatment, reveals that

$$J(3P)=\frac{\Gamma(4/3)^s}{\Gamma(s/3)}n^{s/3-1}. \tag{6.9}$$

Meanwhile, on noting that $sP^3<n$, a second application of Fourier's integral formula leads to the conclusion that

$$J(P)=0. \tag{6.10}$$

The desired formula (6.6) now follows from (6.7)–(6.10).

Now write $\mathfrak{m}=[0,1)\setminus\mathfrak{M}$. Then for $n\in\mathcal{Z}_s(N,M)$, on recalling our implicit hypothesis that $\psi_s(n)=O(n^\delta)$ for some sufficiently small positive number δ, it follows from (1.6), (6.4) and (6.6) that

$$\Big|\sum_{j=1}^{s}(-1)^{j+1}\binom{s}{j}\mathcal{R}_{s,j}(\mathfrak{m})\Big|>\tfrac{1}{2}n^{s/3-1}\psi_s(n)^{-1},$$

whence

$$\sum_{j=1}^{s}\binom{s}{j}|\mathcal{R}_{s,j}(\mathfrak{m})|>\tfrac{1}{2}n^{s/3-1}\psi_s(n)^{-1}. \tag{6.11}$$

Define the complex numbers $\eta_{s,j}(n)$ by taking $\eta_{s,j}(n)=0$ for $n\notin\mathcal{Z}_s(N,M)$, and when $n\in\mathcal{Z}_s(N,M)$ by means of the equation

$$\Big|\int_{\mathfrak{m}} h_1(\alpha)^j h(\alpha)^{s-j}e(-n\alpha)d\alpha\Big|=\eta_{s,j}(n)\int_{\mathfrak{m}} h_1(\alpha)^j h(\alpha)^{s-j}e(-n\alpha)d\alpha.$$

Plainly, one has $|\eta_{s,j}(n)| \le 1$ for every natural number n. In view of (6.5) and (6.11), we deduce that

$$\begin{aligned} N^{s/3-1}\psi_s(N)^{-1}Z_s(N,M) \\ &\ll \sum_{j=1}^{s}\binom{s}{j}\sum_{N<n\le N+M}\eta_{s,j}(n)\int_{\mathfrak{m}} h_1(\alpha)^j h(\alpha)^{s-j}e(-n\alpha)d\alpha \\ &= \sum_{j=1}^{s}\binom{s}{j}\int_{\mathfrak{m}} h_1(\alpha)^j h(\alpha)^{s-j}K_{s,j}(-\alpha)d\alpha, \end{aligned}$$

where

$$K_{s,j}(\alpha) = \sum_{N<n\le N+M}\eta_{s,j}(n)e(n\alpha). \tag{6.12}$$

Thus we conclude that

$$N^{s/3-1}\psi_s(N)^{-1}Z_s(N,M) \ll \max_{1\le j\le s}\int_{\mathfrak{m}} |h_1(\alpha)^j h(\alpha)^{s-j}K_{s,j}(\alpha)|d\alpha. \tag{6.13}$$

Let J be the index for which the maximum is achieved on the right hand side of (6.13). Then by applying Hölder's inequality, one obtains

$$\int_{\mathfrak{m}} |h_1(\alpha)^J h(\alpha)^{s-J}K_{s,J}(\alpha)|d\alpha \le \mathcal{I}_1^{1/2}\mathcal{I}_2^{(J-1)/(2s-2)}\mathcal{I}_3^{(s-J)/(2s-2)}, \tag{6.14}$$

where

$$\mathcal{I}_1 = \int_0^1 |h_1(\alpha)K_{s,J}(\alpha)|^2 d\alpha, \tag{6.15}$$

$$\mathcal{I}_2 = \int_{\mathfrak{m}} |h_1(\alpha)|^{2s-2}d\alpha \quad \text{and} \quad \mathcal{I}_3 = \int_{\mathfrak{m}} |h(\alpha)|^{2s-2}d\alpha.$$

But on recalling that $s = 4 + t$, one finds that

$$\mathcal{I}_2 \le \Big(\sup_{\alpha\in\mathfrak{m}}|h_1(\alpha)|\Big)^{2t-2}\int_{\mathfrak{m}} |h_1(\alpha)|^8 d\alpha. \tag{6.16}$$

The methods of Vaughan [19], as refined by Boklan [2], yield the upper bound

$$\int_{\mathfrak{m}} |h_1(\alpha)|^8 d\alpha \ll P^5(\log P)^{\varepsilon-3},$$

for any positive number ε. Meanwhile, on combining the refined estimates of Hall and Tenenbaum [13] for Hooley's Δ-function with the proof of Lemma 1 of Vaughan [19], one obtains

$$\sup_{\alpha\in\mathfrak{m}}|h_1(\alpha)| \ll P^{3/4}(\log P)^{1/4+\varepsilon}.$$

Thus we conclude from (6.16) that

$$\begin{aligned}\mathcal{I}_2 &\ll (P^{3/4}(\log P)^{1/4+\varepsilon})^{2t-2}P^5(\log P)^{\varepsilon-3}\\ &\ll P^{(3s-5)/2}(\log P)^{\varepsilon+(s-11)/2},\end{aligned}\tag{6.17}$$

and a similar argument shows likewise that

$$\mathcal{I}_3 \ll P^{(3s-5)/2}(\log P)^{\varepsilon+(s-11)/2}.\tag{6.18}$$

In order to estimate the mean value (6.15), we begin by noting, from orthogonality, that it follows from (6.12) that $\mathcal{I}_1$ is bounded above by the number of integral solutions of the equation

$$y_1^3 - y_2^3 = n_1 - n_2,\tag{6.19}$$

with $P < y_i \le 3P$ $(i = 1, 2)$ and $n_l \in \mathcal{Z}_s(N, M)$ $(l = 1, 2)$. But whenever $y_1 \ne y_2$, it follows from the latter conditions that

$$|y_1^3 - y_2^3| > 3P^2 > M \ge |n_1 - n_2|.$$

We are forced to conclude that the only solutions of the equation (6.19) satisfy $y_1 = y_2$ and $n_1 = n_2$, whence

$$\mathcal{I}_1 \ll PZ_s(N, M).\tag{6.20}$$

On substituting (6.17), (6.18) and (6.20) into (6.14), we obtain the estimate

$$\begin{aligned}&\max_{1\le j\le s}\int_{\mathfrak{m}}|h_1(\alpha)^j h(\alpha)^{s-j}K_{s,j}(\alpha)|d\alpha\\ &\qquad\ll (PZ_s)^{1/2}\left(P^{(3s-5)/2}(\log P)^{\varepsilon+(s-11)/2}\right)^{1/2},\end{aligned}$$

whence by (6.13), we have

$$N^{(s-3)/3}\psi_s(N)^{-1}Z_s \ll P^{(3s-3)/4}(\log P)^{\varepsilon+(s-11)/4}Z_s^{1/2}.$$

On recalling that $P = \frac{1}{2}N^{1/3}$, we therefore conclude that

$$Z_s \ll P^{(9-s)/2}(\log P)^{\varepsilon-(11-s)/2}\psi_s(N)^2,$$

and the desired conclusion (6.1) follows immediately. This completes the proof of Theorem 1.3.

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