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The density of integral solutions for pairs of diagonal cubic equations

Jörg Brüdern and Trevor D. Wooley

Abstract. We investigate the number of integral solutions possessed by a pair of diagonal cubic equations in a large box. Provided that the number of variables in the system is at least thirteen, and in addition the number of variables in any non-trivial linear combination of the underlying forms is at least seven, we obtain a lower bound for the order of magnitude of the number of integral solutions consistent with the product of local densities associated with the system.

1. Introduction

This paper is concerned with the solubility in integers of the equations

$$a_1x_1^3 + a_2x_2^3 + \ldots + a_sx_s^3 = b_1x_1^3 + b_2x_2^3 + \ldots + b_sx_s^3 = 0, \tag{1.1}$$

where $(a_i, b_i) \in \mathbb{Z}^2\setminus\{\mathbf{0}\}$ are fixed coefficients. It is natural to enquire to what extent the Hasse principle holds for such systems of equations. Cook [**C85**], refining earlier work of Davenport and Lewis [**DL66**], has analysed the local solubility problem with great care. He showed that when $s \geq 13$ and p is a prime number with $p \neq 7$, then the system (1.1) necessarily possesses a non-trivial solution in $\mathbb{Q}_p$. Here, by *non-trivial solution*, we mean any solution that differs from the obvious one in which $x_j = 0$ for $1 \leq j \leq s$. No such conclusion can be valid for $s \leq 12$, for there may then be local obstructions for any given set of primes p with $p \equiv 1 \pmod 3$; see [**BW06**] for an example that illuminates this observation. The 7-adic case, moreover, is decidedly different. For $s \leq 15$ there may be 7-adic obstructions to the solubility of the system (1.1), and so it is only when $s \geq 16$ that the existence of non-trivial solutions in $\mathbb{Q}_7$ is assured. This much was known to Davenport and Lewis [**DL66**].

Were the Hasse principle to hold for systems of the shape (1.1), then in view of the above discussion concerning the local solubility problem, the existence of

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integer solutions to the equations (1.1) would be decided in $\mathbb{Q}_7$ alone whenever $s \geq 13$. Under the more stringent hypothesis $s \geq 14$, this was confirmed by the first author [**B90**], building upon the efforts of Davenport and Lewis [**DL66**], Cook [**C72**], Vaughan [**V77**] and Baker and Brüdern [**BB88**] spanning an interval of more than twenty years. In a recent collaboration [**BW06**] we have been able to add the elusive case $s = 13$, and may therefore enunciate the following conclusion.

THEOREM 1. *Suppose that $s \geq 13$. Then for any choice of coefficients $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ $(1 \leq j \leq s)$, the simultaneous equations* (1.1) *possess a non-trivial solution in rational integers if and only if they admit a non-trivial solution in $\mathbb{Q}_7$.*

Now let $\mathcal{N}_s(P)$ denote the number of solutions of the system (1.1) in rational integers $x_1, \ldots, x_s$ satisfying the condition $|x_j| \leq P$ $(1 \leq j \leq s)$. When s is large, a naïve application of the philosophy underlying the circle method suggests that $\mathcal{N}_s(P)$ should be of order P^{s-6} in size, but in certain cases this may be false even in the absence of local obstructions. This phenomenon is explained by the failure of the Hasse principle for certain diagonal cubic forms in four variables. When $s \geq 10$ and $b_1, \ldots, b_s \in \mathbb{Z} \setminus \{0\}$, for example, the simultaneous equations

$$5x_1^3 + 9x_2^3 + 10x_3^3 + 12x_4^3 = b_1x_1^3 + b_2x_2^3 + \ldots + b_sx_s^3 = 0 \tag{1.2}$$

have non-trivial (and non-singular) solutions in every p-adic field $\mathbb{Q}_p$ as well as in $\mathbb{R}$, yet all solutions in rational integers must satisfy the condition $x_i = 0$ $(1 \leq i \leq 4)$. The latter must hold, in fact, independently of the number of variables. For such examples, therefore, one has $\mathcal{N}_s(P) = o(P^{s-6})$ when $s \geq 9$, whilst for $s \geq 12$ one may show that $\mathcal{N}_s(P)$ is of order P^{s-7}. For more details, we refer the reader to the discussion surrounding equation (1.2) of [**BW06**]. This example also shows that weak approximation may fail for the system (1.1), even when s is large.

In order to measure the extent to which a system (1.1) may resemble the pathological example (1.2), we introduce the number q_0, which we define by

$$q_0 = \min_{(c,d) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}} \operatorname{card}\{1 \leq j \leq s : ca_j + db_j \neq 0\}.$$

This important invariant of the system (1.1) has the property that as q_0 becomes larger, the counting function $\mathcal{N}_s(P)$ behaves more tamely. Note that in the example (1.2) discussed above one has $q_0 = 4$ whenever $s \geq 8$.

THEOREM 2. *Suppose that $s \geq 13$, and that $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ $(1 \leq j \leq s)$ satisfy the condition that the system* (1.1) *admits a non-trivial solution in $\mathbb{Q}_7$. Then whenever $q_0 \geq 7$, one has $\mathcal{N}_s(P) \gg P^{s-6}$.*

The conclusion of Theorem 2 was obtained in our recent paper [**BW06**] for all cases wherein $q_0 \geq s - 5$. This much suffices to establish Theorem 1; see §8 of [**BW06**] for an account of this deduction. Our main objective in this paper is a detailed discussion of the cases with $7 \leq q_0 \leq s - 6$. We remark that the arguments of this paper as well as those in [**BW06**] extend to establish weak approximation for the system (1.1) when $s \geq 13$ and $q_0 \geq 7$. In the special cases in which $s = 13$ and q_0 is equal to either 5 or 6, a conditional proof of weak approximation is possible by invoking recent work of Swinnerton-Dyer [**SD01**], subject to the as yet unproven finiteness of the Tate-Shafarevich group for elliptic curves over quadratic fields. Indeed, equipped with the latter conclusion, for these particular values of q_0 one may relax the condition on s beyond that addressed by Theorem 2. When $s = 13$

and $q_0 \le 4$, on the other hand, weak approximation fails in general, as we have already seen in the discussion accompanying the system (1.2).

The critical input into the proof of Theorem 2 is a certain arithmetic variant of Bessel's inequality established in [**BW06**]. We begin in §2 by briefly sketching the principal ideas underlying this innovation. In §3 we prepare the ground for an application of the Hardy-Littlewood method, deriving a lower bound for the major arc contribution in the problem at hand. Some delicate footwork in §4 establishes a mean value estimate that, in all circumstances save for particularly pathological situations, leads in §5 to a viable complementary minor arc estimate sufficient to establish Theorem 2. The latter elusive situations are handled in §6 via an argument motivated by our recent collaboration [**BKW01a**] with Kawada, and thereby we complete the proof of Theorem 2. Finally, in §7, we make some remarks concerning the extent to which our methods are applicable to systems containing fewer than 13 variables.

Throughout, the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. In an effort to simplify our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$ the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, from time to time we make use of vector notation in order to save space. Thus, for example, we may abbreviate $(c_1, \dots, c_t)$ to $\mathbf{c}$.

2. An arithmetic variant of Bessel's inequality

The major innovation in our earlier paper [**BW06**] is an arithmetic variant of Bessel's inequality that sometimes provides good mean square estimates for Fourier coefficients averaged over sparse sequences. Since this tool plays a crucial role also in our current excursion, we briefly sketch the principal ideas. When P and R are real numbers with $1 \le R \le P$, we define the set of smooth numbers $\mathcal{A}(P,R)$ by

$$\mathcal{A}(P,R) = \{n \in \mathbb{N} \cap [1,P] \,:\, p \text{ prime and } p|n \Rightarrow p \le R\}.$$

The Fourier coefficients that are to be averaged arise in connection with the smooth cubic Weyl sum $h(\alpha) = h(\alpha; P, R)$, defined by

$$h(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^3), \tag{2.1}$$

where here and later we write $e(z)$ for $\exp(2\pi i z)$. The sixth moment of this sum has played an important role in many applications in recent years, and that at hand is no exception to the rule. Write $\xi = (\sqrt{2833} - 43)/41$. Then as a consequence of the work of the second author [**W00**], given any positive number ε, there exists a positive number $\eta = \eta(\varepsilon)$ with the property that whenever $1 \le R \le P^\eta$, one has

$$\int_0^1 |h(\alpha; P, R)|^6 \, d\alpha \ll P^{3+\xi+\varepsilon}. \tag{2.2}$$

We assume henceforth that whenever R appears in a statement, either implicitly or explicitly, then $1 \le R \le P^\eta$ with η a positive number sufficiently small in the context of the upper bound (2.2).

The Fourier coefficients over which we intend to average are now defined by

$$\psi(n) = \int_0^1 |h(\alpha)|^5 e(-n\alpha)\, d\alpha. \tag{2.3}$$

An application of Parseval's identity in combination with conventional circle method technology readily shows that $\sum_n \psi(n)^2$ is of order P^7. Rather than average $\psi(n)$ in mean square over all integers, we instead restrict to the sparse sequence consisting of differences of two cubes, and establish the bound

$$\sum_{1\le x,y\le P} \psi(x^3-y^3)^2 \ll P^{6+\xi+4\varepsilon}. \tag{2.4}$$

Certain contributions to the sum on the left hand side of (2.4) are easily estimated. By Hua's Lemma (see Lemma 2.5 of [**V97**]) and a consideration of the underlying Diophantine equations, one has

$$\int_0^1 |h(\alpha)|^4\, d\alpha \ll P^{2+\varepsilon}.$$

On applying Schwarz's inequality to (2.3), we therefore deduce from (2.2) that the estimate $\psi(n) = O(P^{5/2+\xi/2+\varepsilon})$ holds uniformly in n. We apply this upper bound with $n=0$ in order to show that the terms with $x=y$ contribute at most $O(P^{6+\xi+2\varepsilon})$ to the left hand side of (2.4). The integers x and y with $1\le x,y\le P$ and $|\psi(x^3-y^3)|\le P^{2+\xi/2+2\varepsilon}$ likewise contribute at most $O(P^{6+\xi+4\varepsilon})$ within the summation of (2.4). We estimate the contribution of the remaining Fourier coefficients by dividing into dyadic intervals. When T is a real number with

$$P^{2+\xi/2+2\varepsilon} \le T \le P^{5/2+\xi/2+2\varepsilon}, \tag{2.5}$$

define $\mathcal{Z}(T)$ to be the set of ordered pairs $(x,y)\in\mathbb{N}^2$ with

$$1\le x,y\le P,\quad x\ne y \quad\text{and}\quad T\le |\psi(x^3-y^3)|\le 2T, \tag{2.6}$$

and write $Z(T)$ for $\operatorname{card}(\mathcal{Z}(T))$. Then on incorporating in addition the contributions of those terms already estimated, a familiar dissection argument now demonstrates that there is a number T satisfying (2.5) for which

$$\sum_{1\le x,y\le P} \psi(x^3-y^3)^2 \ll P^{6+\xi+4\varepsilon} + P^{\varepsilon}T^2 Z(T). \tag{2.7}$$

An upper bound for $Z(T)$ at this point being all that is required to complete the proof of the estimate (2.4), we set up a mechanism for deriving such an upper bound that has its origins in work of Brüdern, Kawada and Wooley [**BKW01a**] and Wooley [**W02**]. Let $\sigma(n)$ denote the sign of the real number $\psi(n)$ defined in (2.3), with the convention that $\sigma(n)=0$ when $\psi(n)=0$, so that $\psi(n)=\sigma(n)|\psi(n)|$. Then on forming the exponential sum

$$K_T(\alpha) = \sum_{(x,y)\in\mathcal{Z}(T)} \sigma(x^3-y^3)e(\alpha(y^3-x^3)),$$

we find from (2.3) and (2.6) that

$$\int_0^1 |h(\alpha)|^5 K_T(\alpha)\, d\alpha \ge TZ(T).$$

An application of Schwarz's inequality in combination with the upper bound (2.2) therefore permits us to infer that

$$TZ(T) \ll (P^{3+\xi+\varepsilon})^{1/2}\Big(\int_0^1 |h(\alpha)^4 K_T(\alpha)^2|d\alpha\Big)^{1/2}. \tag{2.8}$$

Next, on applying Weyl's differencing lemma (see, for example, Lemma 2.3 of [**V97**]), one finds that for certain non-negative numbers t_l, satisfying $t_l = O(P^\varepsilon)$ for $0 < |l| \le P^3$, one has

$$|h(\alpha)|^4 \ll P^3 + P \sum_{0<|l|\le P^3} t_l e(\alpha l).$$

Consequently, by orthogonality,

$$\begin{aligned}\int_0^1 |h(\alpha)^4 K_T(\alpha)^2|\, d\alpha &\ll P^3 \int_0^1 |K_T(\alpha)|^2 d\alpha + P^{1+\varepsilon} K_T(0)^2 \\ &\ll P^\varepsilon (P^3 Z(T) + PZ(T)^2).\end{aligned}$$

Here we have applied the simple fact that when m is a non-zero integer, the number of solutions of the Diophantine equation $m = x^3 - y^3$ with $1 \le x, y \le P$ is at most $O(P^\varepsilon)$. Since $T \ge P^{2+\xi/2+2\varepsilon}$, the upper bound $Z(T) = O(T^{-2}P^{6+\xi+2\varepsilon})$ now follows from the relation (2.8). On substituting the latter estimate into (2.7), the desired conclusion (2.4) is now immediate.

Note that in the summation on the left hand side of the estimate (2.4), one may restrict the summation over the integers x and y to any subset of $[1, P]^2$ without affecting the right hand side. Thus, on recalling the definition (2.3), we see that we have proved the special case $a = b = c = d = 1$ of the following lemma.

Lemma 3. *Let a, b, c, d denote non-zero integers. Then for any subset $\mathcal{B}$ of $[1, P] \cap \mathbb{Z}$, one has*

$$\int_0^1\int_0^1 |h(a\alpha)h(b\beta)|^5 \Big|\sum_{x\in\mathcal{B}} e((c\alpha + d\beta)x^3)\Big|^2 d\alpha\, d\beta \ll P^{6+\xi+\varepsilon}.$$

This lemma is a restatement of Theorem 3 of [**BW06**]. It transpires that no great difficulty is encountered when incorporating the coefficients a, b, c, d into the argument described above; see §3 of [**BW06**].

We apply Lemma 3 in the cosmetically more general formulation provided by the following lemma.

Lemma 4. *Suppose that c_i, d_i $(1 \le i \le 3)$ are integers satisfying the condition*

$$(c_1d_2 - c_2d_1)(c_1d_3 - c_3d_1)(c_2d_3 - c_3d_2) \neq 0.$$

Write $\lambda_j = c_j\alpha + d_j\beta$ $(j = 1, 2, 3)$. Then for any subset $\mathcal{B}$ of $[1, P] \cap \mathbb{Z}$, one has

$$\int_0^1\int_0^1 |h(\lambda_1)h(\lambda_2)|^5 \Big|\sum_{x\in\mathcal{B}} e(\lambda_3 x^3)\Big|^2 d\alpha\, d\beta \ll P^{6+\xi+\varepsilon}.$$

Proof. The desired conclusion follows immediately from Lemma 3 on making a change of variable. The reader may care to compare the situation here with that occurring in the estimation of the integral J_3 in the proof of Theorem 4 of [**BW06**] (see §4 of the latter). □

3. Preparation for the circle method

The next three sections of this paper are devoted to the proof of Theorem 2. In view of the hypotheses of the theorem together with the discussion following its statement, we may suppose henceforth that $s \geq 13$ and $7 \leq q_0 \leq s-6$. With the pairs $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ $(1 \leq j \leq s)$, we associate both the linear forms

$$\Lambda_j = a_j\alpha + b_j\beta \quad (1 \leq j \leq s), \tag{3.1}$$

and the two linear forms $L_1(\boldsymbol{\theta})$ and $L_2(\boldsymbol{\theta})$ defined for $\boldsymbol{\theta} \in \mathbb{R}^s$ by

$$L_1(\boldsymbol{\theta}) = \sum_{j=1}^{s} a_j\theta_j \quad \text{and} \quad L_2(\boldsymbol{\theta}) = \sum_{j=1}^{s} b_j\theta_j. \tag{3.2}$$

We say that two forms Λ_i and Λ_j are *equivalent* when there exists a non-zero rational number λ with $\Lambda_i = \lambda\Lambda_j$. This notion defines an equivalence relation on the set $\{\Lambda_1, \Lambda_2, \dots, \Lambda_s\}$, and we refer to the number of elements in the equivalence class $[\Lambda_j]$ containing the form Λ_j as its *multiplicity*. Suppose that the s forms Λ_j $(1 \leq j \leq s)$ fall into T equivalence classes, and that the multiplicities of the representatives of these classes are $R_1, \dots, R_T$. By relabelling variables if necessary, there is no loss in supposing that $R_1 \geq R_2 \geq \dots \geq R_T \geq 1$. Further, by our hypothesis that $7 \leq q_0 \leq s-6$, it is apparent that for any pair $(c,d) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, the linear form $cL_1(\boldsymbol{\theta}) + dL_2(\boldsymbol{\theta})$ necessarily possesses at least 7 non-zero coefficients, and for some choice $(c,d) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ this linear form has at most $s-6$ non-zero coefficients. Thus we may assume without loss of generality that $6 \leq R_1 \leq s-7$.

We distinguish three cases according to the number of variables and the arrangement of the multiplicities of the forms. We refer to a system (1.1) as being of type I when $T = 2$, as being of type II when $T = 3$ and $R_3 = 1$, and as being of type III in the remaining cases wherein $T \geq 3$ and $s - R_1 - R_2 \geq 2$. The argument required to address the systems of types I and II is entirely different from that required for those of type III, and we defer an account of these former situations to §6 below. Our purpose in the remainder of §3 together with §§4 and 5 is to establish the conclusion of Theorem 2 for type III systems.

Consider then a type III system (1.1) with $s \geq 13$ and $7 \leq q_0 \leq s-6$, and consider a fixed subset $\mathcal{S}$ of $\{1, \dots, s\}$ with $\operatorname{card}(\mathcal{S}) = 13$. We may suppose that the 13 forms Λ_j $(j \in \mathcal{S})$ fall into t equivalence classes, and that the multiplicities of the representatives of these classes are $r_1, \dots, r_t$. By relabelling variables if necessary, there is no loss in supposing that $r_1 \geq r_2 \geq \dots \geq r_t \geq 1$. The condition $R_1 \leq s-7$ ensures that $R_2 + R_3 + \dots + R_T \geq 7$. Thus, on recalling the additional conditions $s \geq 13$, $T \geq 3$, $R_1 \geq 6$ and $s - R_1 - R_2 \geq 2$, it is apparent that we may make a choice for $\mathcal{S}$ in such a manner that $t \geq 3$, $r_1 = 6$ and $13 - r_1 - r_2 \geq 2$. We may therefore suppose that the profile of multiplicities $(r_1, r_2, \dots, r_t)$ satisfies $t \geq 3$, $r_1 = 6$, $r_2 \leq 5$ and $r_2 + r_3 + \dots + r_t = 7$. But then, in view of our earlier condition $r_1 \geq r_2 \geq \dots \geq r_t \geq 1$, we find that necessarily $r_t \leq 3$. We now relabel variables in the system (1.1), and likewise in (3.1) and (3.2), so that the set $\mathcal{S}$ becomes $\{1, 2, \dots, 13\}$, and so that Λ_1 becomes a form in the first equivalence class counted by r_1, so that Λ_2 becomes a form in the second equivalence class counted by r_2, and Λ_{13} becomes a form in the tth equivalence class counted by r_t.

We next make some simplifying transformations that ease the analysis of the singular integral, and here we follow the pattern of our earlier work [**BW06**]. First, by taking suitable integral linear combinations of the equations (1.1), we may suppose without loss that

$$b_1 = a_2 = 0 \quad \text{and} \quad b_i = 0 \qquad (8 \le i \le 12). \tag{3.3}$$

Since we may suppose that $a_1 b_2 \neq 0$, the simultaneous equations

$$L_1(\boldsymbol{\theta}) = L_2(\boldsymbol{\theta}) = 0 \tag{3.4}$$

possess a solution $\boldsymbol{\theta}$ with $\theta_j \neq 0$ $(1 \le j \le s)$. Applying the substitution $x_j \to -x_j$ for those indices j with $1 \le j \le s$ for which $\theta_j < 0$, neither the solubility of the system (1.1), nor the corresponding function $\mathcal{N}_s(P)$, are affected, yet the transformed linear system associated with (3.4) has a solution $\boldsymbol{\theta}$ with $\theta_j > 0$ $(1 \le j \le s)$. In addition, the homogeneity of the system (3.4) ensures that a solution of the latter type may be chosen with $\boldsymbol{\theta} \in (0,1)^s$. We now fix this solution $\boldsymbol{\theta}$, and fix also ε to be a sufficiently small positive number, and η to be a positive number sufficiently small in the context of Lemmata 3 and 4 with the property that $\boldsymbol{\theta} \in (\eta, 1)^s$.

At this point we are ready to define the generating functions required in our application of the circle method. In addition to the smooth Weyl sum $h(\alpha)$ defined in (2.1) we require also the classical Weyl sum

$$g(\alpha) = \sum_{\eta P < x \le P} e(\alpha x^3).$$

On defining the generating functions

$$H(\alpha,\beta) = \prod_{j=2}^{12} h(\Lambda_j) \quad \text{and} \quad G(\alpha,\beta) = \prod_{j=13}^{s} g(\Lambda_j), \tag{3.5}$$

we now see from orthogonality that

$$\mathcal{N}_s(P) \ge \int_0^1 \int_0^1 g(\Lambda_1) H(\alpha,\beta) G(\alpha,\beta)\, d\alpha\, d\beta. \tag{3.6}$$

We apply the circle method to obtain a lower bound for the integral on the right hand side of (3.6). In this context, we put $Q = (\log P)^{1/100}$, and when $a, b \in \mathbb{Z}$ and $q \in \mathbb{N}$, we write

$$\mathfrak{N}(q,a,b) = \{(\alpha,\beta) \in [0,1)^2 \,:\, |\alpha - a/q| \le QP^{-3} \text{ and } |\beta - b/q| \le QP^{-3}\}.$$

We then define the major arcs $\mathfrak{N}$ of our Hardy-Littlewood dissection to be the union of the sets $\mathfrak{N}(q,a,b)$ with $0 \le a, b \le q \le Q$ and $(q,a,b) = 1$. The corresponding set $\mathfrak{n}$ of minor arcs are defined by $\mathfrak{n} = [0,1)^2 \setminus \mathfrak{N}$.

It transpires that the contribution of the major arcs within the integral on the right hand side of (3.6) is easily estimated by making use of the work from our previous paper [**BW06**].

Lemma 5. *Suppose that the system* (1.1) *is of type III with* $s \ge 13$ *and* $7 \le q_0 \le s - 6$*, and possesses a non-trivial* 7*-adic solution. Then, in the setting described in the prequel, one has*

$$\iint_{\mathfrak{N}} g(\Lambda_1) H(\alpha,\beta) G(\alpha,\beta)\, d\alpha\, d\beta \gg P^{s-6}.$$

Proof. Although the formulation of the statements of Lemmata 12 and 13 of [**BW06**] may appear more restrictive than our present circumstances permit, an examination of their proofs will confirm that it is sufficient in fact that the maximum multiplicity of any of $\Lambda_1, \Lambda_2, \dots, \Lambda_{13}$ is at most six amongst the latter forms. Such follows already from the hypotheses of the lemma at hand, and thus the desired conclusion follows in all essentials from the estimate (7.8) of [**BW06**] together with the conclusions of Lemmata 12 and 13 of the latter paper. Note that in [**BW06**] the generating functions employed differ slightly from those herein, in that the exponential sums corresponding to the forms $\Lambda_{13}, \dots, \Lambda_s$ are smooth Weyl sums rather than the present classical Weyl sums. This deviation, however, demands at most cosmetic alterations to the argument of §7 of [**BW06**], and we spare the reader the details. It should be remarked, though, that it is the reference to Lemma 13 of [**BW06**] that calls for the specific construction of the point $\boldsymbol{\theta}$ associated with the equations (3.4). $\square$

4. The auxiliary mean value estimate

The estimate underpinning our earlier work [**BW06**] takes the shape

$$\int_0^1 \int_0^1 |h(\Lambda_1)h(\Lambda_2)\dots h(\Lambda_{12})|\, d\alpha\, d\beta \ll P^{6+\xi+\varepsilon},$$

predicated on the assumption that the maximum multiplicity amongst $\Lambda_1, \dots, \Lambda_{12}$ does not exceed 5. In order to make progress on a viable minor arc treatment in the present situation, we require an analogue of this estimate that permits the replacement of a smooth Weyl sum by a corresponding classical Weyl sum. In preparation for this lemma, we recall an elementary observation from our earlier work, the proof of which is almost self-evident (see Lemma 5 of [**BW06**]).

Lemma 6. *Let k and N be natural numbers, and suppose that $\mathfrak{B} \subseteq \mathbb{C}^k$ is measurable. Let $\omega_i(\mathbf{z})$ $(0 \le i \le N)$ be complex-valued functions of $\mathfrak{B}$. Then whenever the functions $|\omega_0(\mathbf{z})\omega_j(\mathbf{z})^N|$ $(1 \le j \le N)$ are integrable on $\mathfrak{B}$, one has the upper bound*

$$\int_{\mathfrak{B}} |\omega_0(\mathbf{z})\omega_1(\mathbf{z})\dots\omega_N(\mathbf{z})|\, d\mathbf{z} \le N \max_{1\le j\le N} \int_{\mathfrak{B}} |\omega_0(\mathbf{z})\omega_j(\mathbf{z})^N|\, d\mathbf{z}.$$

It is convenient in what follows to abbreviate, for each index l, the expression $|h(\Lambda_l)|$ simply to h_l, and likewise $|g(\Lambda_l)|$ to g_l and $|G(\alpha,\beta)|$ to G. Furthermore, we write

$$G_0(\alpha,\beta) = \prod_{j=14}^{s} g(\Lambda_j), \tag{4.1}$$

with the implicit convention that $G_0(\alpha,\beta)$ is identically 1 when $s < 14$.

Lemma 7. *Suppose that the system* (1.1) *is of type III with $s \ge 13$ and $7 \le q_0 \le s-6$. Then in the setting described in* §3, *one has*

$$\int_0^1 \int_0^1 |H(\alpha,\beta)G(\alpha,\beta)|\, d\alpha\, d\beta \ll P^{s-7+\xi+\varepsilon}.$$

Proof. We begin by making some analytic observations that greatly simplify the combinatorial details of the argument to come. Write $\mathcal{L} = \{\Lambda_2, \Lambda_3, \dots, \Lambda_{12}\}$, and suppose that the number of equivalence classes in $\mathcal{L}$ is u. By relabelling indices if necessary, we may suppose that $u \ge 3$ and that representatives of these classes are $\widetilde{\Lambda}_i \in \mathcal{L}$ $(1 \le i \le u)$. For each index i we denote by s_i the multiplicity of $\widetilde{\Lambda}_i$ amongst the elements of the set $\mathcal{L}$. Then according to the discussion of the previous section, we may suppose that $\Lambda_1 \in [\widetilde{\Lambda}_1]$, that

$$1 \le s_u \le s_{u-1} \le \ldots \le s_1 = 5 \quad \text{and} \quad s_2 + s_3 + \cdots + s_u = 6, \tag{4.2}$$

and further that if $\Lambda_{13} \in [\widetilde{\Lambda}_i]$ for some index i with $1 \le i \le u$, then in fact

$$\Lambda_{13} \in [\widetilde{\Lambda}_u] \quad \text{and} \quad 1 \le s_u \le 2. \tag{4.3}$$

Next, for a given index i with $2 \le i \le 12$, consider the linear forms Λ_{l_j} $(1 \le j \le s_i)$ equivalent to Λ_i from the set $\mathcal{L}$. Apply Lemma 6 with $N = s_i$, with h_{l_j} in place of ω_j $(1 \le j \le N)$, and with ω_0 replaced by the product of those h_l with $\Lambda_l \notin [\widetilde{\Lambda}_i]$ $(2 \le l \le 12)$, multiplied by $G(\alpha, \beta)$. Then it is apparent that there is no loss of generality in supposing that $\Lambda_{l_j} = \widetilde{\Lambda}_i$ $(1 \le j \le s_i)$. By repeating this argument for successive equivalence classes, moreover, we find that a suitable choice of equivalence class representatives $\widetilde{\Lambda}_l$ $(1 \le l \le u)$ yields the bound

$$\int_0^1 \int_0^1 |H(\alpha, \beta) G(\alpha, \beta)| \, d\alpha \, d\beta \ll \int_0^1 \int_0^1 G \tilde{h}_1^{s_1} \tilde{h}_2^{s_2} \ldots \tilde{h}_u^{s_u} \, d\alpha \, d\beta, \tag{4.4}$$

where we now take the liberty of abbreviating $|h(\widetilde{\Lambda}_l)|$ simply to $\tilde{h}_l$ for each l.

A further simplification is achieved through the use of a device employed in the proof of Lemma 6 of [**BW06**]. We begin by considering the situation in which $\Lambda_{13} \in [\widetilde{\Lambda}_u]$. Let ν be a non-negative integer, and suppose that $s_{u-2} = s_{u-1} + \nu < 5$. Then we may apply Lemma 6 with $N = \nu + 2$, with $\tilde{h}_{u-2}$ in place of ω_i $(1 \le i \le \nu+1)$ and $\tilde{h}_{u-1}$ in place of ω_N, and with ω_0 set equal to

$$G \tilde{h}_1^{s_1} \tilde{h}_2^{s_2} \ldots \tilde{h}_{u-3}^{s_{u-3}} \tilde{h}_{u-2}^{s_{u-2}-\nu-1} \tilde{h}_{u-1}^{s_{u-1}-1} \tilde{h}_u^{s_u}.$$

Here, and in what follows, we interpret the vanishing of any exponent as indicating that the associated exponential sum is deleted from the product. In this way we obtain an upper bound of the shape (4.4) in which the exponents s_{u-2} and $s_{u-1} = s_{u-2} - \nu$ are replaced by $s_{u-2} + 1$ and $s_{u-1} - 1$, respectively, or else by $s_{u-2} - \nu - 1$ and $s_{u-1} + \nu + 1$. By relabelling if necessary, we derive an upper bound of the shape (4.4), subject to the constraints (4.2) and (4.3), wherein either the parameter s_{u-1} is reduced, or else the parameter u is reduced. By repeating this process, therefore, we ultimately arrive at a situation in which $u = 3$ and $s_{u-1} = 6 - s_u$, and then the constraints (4.2) and (4.3) imply that necessarily $(s_1, s_2, \ldots, s_u) = (5, 6 - s_3, s_3)$ with $s_3 = 1$ or 2. When $\Lambda_{13} \notin [\widetilde{\Lambda}_u]$ we may proceed likewise, but in the above argument s_{u-1} now plays the rôle of s_{u-2}, and s_u that of s_{u-1}, and with concommitant adjustments to the associated indices throughout. In this second situation we ultimately arrive at a scenario in which $u = 3$ and $s_{u-1} = 5$, and in these circumstances the constraints (4.2) imply that necessarily $(s_1, s_2, \ldots, s_u) = (5, 5, 1)$.

On recalling (4.1) and (4.4), and making use of a trivial inequality for $|G_0(\alpha, \beta)|$, we may conclude thus far that

$$\int_0^1 \int_0^1 |H(\alpha, \beta) G(\alpha, \beta)| \, d\alpha \, d\beta \ll P^{s-13} \int_0^1 \int_0^1 g_{13} \tilde{h}_1^{s_1} \tilde{h}_2^{s_2} \tilde{h}_3^{s_3} \, d\alpha \, d\beta, \tag{4.5}$$

with $(s_1,s_2,s_3)=(5,5,1)$ or $(5,4,2)$. We now write

$$\mathcal{I}_{ij}(\psi)=\int_0^1\int_0^1 \tilde{h}_i^5\tilde{h}_j^5\psi^2\,d\alpha\,d\beta,$$

and we observe that an application of Hölder's inequality yields

$$\int_0^1\int_0^1 g_{13}\tilde{h}_1^{s_1}\tilde{h}_2^{s_2}\tilde{h}_3^{s_3}\,d\alpha\,d\beta\le \mathcal{I}_{12}(g_{13})^{\omega_1}\mathcal{I}_{12}(\tilde{h}_3)^{\omega_2}\mathcal{I}_{13}(\tilde{h}_2)^{\omega_3}, \tag{4.6}$$

where

$$(\omega_1,\omega_2,\omega_3)=\begin{cases}(1/2,1/2,0), & \text{when } s_3=1,\\ (1/2,1/6,1/3), & \text{when } s_3=2.\end{cases}$$

But Lemma 4 is applicable to each of the mean values $\mathcal{I}_{12}(g_{13})$, $\mathcal{I}_{12}(\tilde{h}_3)$ and $\mathcal{I}_{13}(\tilde{h}_2)$, and so we see from (4.6) that

$$\int_0^1\int_0^1 g_{13}\tilde{h}_1^{s_1}\tilde{h}_2^{s_2}\tilde{h}_3^{s_3}\,d\alpha\,d\beta\ll P^{6+\xi+\varepsilon}.$$

The conclusion of Lemma 7 is now immediate on substituting the latter estimate into (4.5). □

5. Minor arcs, with some pruning

Equipped with the mean value estimate provided by Lemma 7, an advance on the minor arc bound complementary to the major arc estimate of Lemma 5 is feasible by the use of appropriate pruning technology. Here, in certain respects, the situation is a little more delicate than was the case in our treatment of the analogous situation in [**BW06**]. The explanation is to be found in the higher multiplicity of coefficient ratios permitted in our present discussion, associated with which is a lower average level of independence amongst the available generating functions.

We begin our account of the minor arcs by defining a set of auxiliary arcs to be employed in the pruning process. Given a parameter X with $1\le X\le P$, we define $\mathfrak{M}(X)$ to be the set of real numbers α with $\alpha\in[0,1)$ for which there exist $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfying $0\le a\le q\le X$, $(a,q)=1$ and $|q\alpha-a|\le XP^{-3}$. We then define sets of major arcs $\mathfrak{M}=\mathfrak{M}(P^{3/4})$ and $\mathfrak{K}=\mathfrak{M}(Q^{1/4})$, and write also $\mathfrak{m}=[0,1)\setminus\mathfrak{M}$ and $\mathfrak{k}=[0,1)\setminus\mathfrak{K}$ for the corresponding sets of minor arcs.

Given a measurable set $\mathfrak{B}\subseteq\mathbb{R}^2$, define the mean-value $\mathcal{J}(\mathfrak{B})$ by

$$\mathcal{J}(\mathfrak{B})=\iint_{\mathfrak{B}}|g(a_1\alpha)G(\alpha,\beta)H(\alpha,\beta)|\,d\alpha\,d\beta. \tag{5.1}$$

Also, put $\mathfrak{E}=\{(\alpha,\beta)\in\mathfrak{n}:\alpha\in\mathfrak{M}\}$. Then on recalling the enhanced version of Weyl's inequality afforded by Lemma 1 of Vaughan [**V86**], one finds from Lemma 7 that

$$\begin{aligned}\mathcal{J}(\mathfrak{n})&\ll\mathcal{J}(\mathfrak{E})+\sup_{\alpha\in\mathfrak{m}}|g(a_1\alpha)|\int_0^1\int_0^1|G(\alpha,\beta)H(\alpha,\beta)|\,d\alpha\,d\beta\\ &\ll\mathcal{J}(\mathfrak{E})+P^{s-6-\tau},\end{aligned} \tag{5.2}$$

wherein we have written

$$\tau=(1/4-\xi)/3. \tag{5.3}$$

Our aim now is to show that $\mathcal{J}(\mathfrak{E}) = o(P^{s-6})$, for then it follows from (5.1) and (5.2) in combination with the conclusion of Lemma 5 that

$$\int_0^1\int_0^1 g(\Lambda_1)G(\alpha,\beta)H(\alpha,\beta)\,d\alpha\,d\beta = \mathcal{J}(\mathfrak{n}) + \iint_{\mathfrak{N}} g(\Lambda_1)G(\alpha,\beta)H(\alpha,\beta)\,d\alpha\,d\beta$$
$$\gg P^{s-6} + o(P^{s-6}).$$

The conclusion $\mathcal{N}_s(P) \gg P^{s-6}$ is now immediate, and this completes the proof of Theorem 2 for systems (1.1) of type III.

Before proceeding further, we define

$$H_0(\alpha,\beta) = \prod_{j=2}^{7} h(\Lambda_j) \quad \text{and} \quad H_1(\alpha) = \prod_{j=8}^{12} h(\Lambda_j), \tag{5.4}$$

wherein we have implicitly made use of the discussion of §3 leading to (3.3) that permits us to assume that $\Lambda_j = a_j\alpha$ $(8 \le j \le 12)$. Also, given $\alpha \in \mathfrak{M}$ we put $\mathfrak{E}(\alpha) = \{\beta \in [0,1) : (\alpha,\beta) \in \mathfrak{E}\}$ and write

$$\Theta(\alpha) = \int_{\mathfrak{E}(\alpha)} |G(\alpha,\beta)H_0(\alpha,\beta)|\,d\beta. \tag{5.5}$$

The relation

$$\mathcal{J}(\mathfrak{E}) = \int_{\mathfrak{M}} |g(a_1\alpha)H_1(\alpha)|\Theta(\alpha)\,d\alpha, \tag{5.6}$$

then follows from (5.1), and it is from here that we launch our pruning argument.

Lemma 8. *One has*

$$\sup_{\alpha\in[0,1)} \Theta(\alpha) \ll P^{s-9} \quad \textit{and} \quad \sup_{\alpha\in\mathfrak{K}} \Theta(\alpha) \ll P^{s-9}Q^{-1/72}.$$

Proof. We divide the set $\mathfrak{E}(\alpha)$ into pieces on which major arc and minor arc estimates of various types may be employed so as to estimate the integral defining $\Theta(\alpha)$ in (5.5). Let $\mathfrak{E}_1(\alpha)$ denote the set consisting of those values β in $\mathfrak{E}(\alpha)$ for which $|g(\Lambda_{13})| < P^{3/4+\tau}$, where τ is defined as in (5.3), and put $\mathfrak{E}_2(\alpha) = \mathfrak{E}(\alpha) \setminus \mathfrak{E}_1(\alpha)$. Then on applying a trivial estimate for those exponential sums $g(\Lambda_j)$ with $j \ge 14$, it follows from (3.5) that

$$\sup_{\beta\in\mathfrak{E}_1(\alpha)} |G(\alpha,\beta)| \ll P^{s-49/4+\tau}. \tag{5.7}$$

But the discussion of §3 leading to (3.3) ensures that $b_j \ne 0$ for $2 \le j \le 7$. By making use of the mean value estimate (2.2), one therefore obtains the estimate

$$\int_0^1 |h(\Lambda_j)|^6\,d\beta = \int_0^1 |h(\gamma)|^6\,d\gamma \ll P^{3+\xi+\varepsilon} \qquad (2 \le j \le 7),$$

whence an application of Hölder's inequality leads from (5.4) to the bound

$$\int_0^1 |H_0(\alpha,\beta)|\,d\beta \le \prod_{j=2}^{7}\Big(\int_0^1 |h(\Lambda_j)|^6\,d\beta\Big)^{1/6} \ll P^{3+\xi+\varepsilon}. \tag{5.8}$$

Consequently, by combining (5.7) and (5.8) we obtain

$$\int_{\mathfrak{E}_1(\alpha)} |G(\alpha,\beta)H_0(\alpha,\beta)|\,d\beta \ll P^{s-9+(\xi-1/4)+\tau+\varepsilon} \ll P^{s-9-\tau}.$$

When $\beta \in \mathfrak{E}_2(\alpha)$, on the other hand, one has $|g(\Lambda_{13})| \geq P^{3/4+\tau}$. Applying the enhanced version of Weyl's inequality already cited, we find that the latter can hold only when $\Lambda_{13} \in \mathfrak{M} \pmod 1$. If we now define the set $\mathfrak{F}(\alpha)$ by

$$\mathfrak{F}(\alpha) = \{\beta \in [0,1) : (\alpha,\beta) \in \mathfrak{n} \text{ and } \Lambda_{13} \in \mathfrak{M} \pmod 1\},$$

and apply a trivial estimate once again for $g(\Lambda_j)$ $(j \geq 14)$, then we may summarise our deliberations thus far with the estimate

$$\Theta(\alpha) \ll P^{s-9-\tau} + P^{s-13} \int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13}) H_0(\alpha,\beta)| \, d\beta. \tag{5.9}$$

A transparent application of Lemma 6 leads from (5.4) to the upper bound

$$\int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13}) H_0(\alpha,\beta)| d\beta \ll \max_{2\leq j\leq 7} \int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13}) h(\Lambda_j)^6| d\beta.$$

The conclusion of the lemma will therefore follow from (5.9) provided that we establish for $2 \leq j \leq 7$ the two estimates

$$\sup_{\alpha\in[0,1)} \int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13}) h(\Lambda_j)^6| \, d\beta \ll P^4 \tag{5.10}$$

and

$$\sup_{\alpha\in\mathfrak{K}} \int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13}) h(\Lambda_j)^6| \, d\beta \ll P^4 Q^{-1/72}. \tag{5.11}$$

We henceforth suppose that j is an index with $2 \leq j \leq 7$, and we begin by considering the upper bound (5.10). Given $\alpha \in [0,1)$, we make the change of variable defined by the substitution $b_{13}\gamma = a_{13}\alpha + b_{13}\beta$. Let $\mathfrak{M}_0$ be defined by

$$\mathfrak{M}_0 = \{\gamma \in [0,1) : b_{13}\gamma \in \mathfrak{M} \pmod 1\}.$$

Then by the periodicity of the integrand modulo 1, the aforementioned change of variable leads to the upper bound

$$\int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13}) h(\Lambda_j)^6| \, d\beta \leq \Big(\sup_{\beta\in\mathfrak{F}(\alpha)} |g(\Lambda_{13})| \Big)^{1/6} \sup_{\lambda\in\mathbb{R}} \mathcal{U}(\lambda), \tag{5.12}$$

in which we write

$$\mathcal{U}(\lambda) = \int_{\mathfrak{M}_0} |g(b_{13}\gamma)|^{5/6} |h(b_j\gamma + \lambda)|^6 \, d\gamma. \tag{5.13}$$

We next examine the first factor on the right hand side of (5.12). Given $\alpha \in \mathfrak{K}$, consider a real number β with $\beta \in \mathfrak{F}(\alpha)$. If it were the case that $\Lambda_{13} \in \mathfrak{K} \pmod 1$, then one would have $\beta = b_{13}^{-1}(\Lambda_{13} - a_{13}\alpha) \in \mathfrak{M}(Q^{3/4})$, whence $(\alpha,\beta) \in \mathfrak{N}$ (see the proof of Lemma 10 in §6 of [**BW06**] for details of a similar argument). But the latter contradicts the hypothesis $\beta \in \mathfrak{F}(\alpha)$, in view of the definition of $\mathfrak{F}(\alpha)$. Thus we conclude that $\Lambda_{13} \in \mathfrak{k} \pmod 1$, and so a standard application of Weyl's inequality (see Lemma 2.4 of [**V97**]) in combination with available major arc estimates (see Theorem 4.1 and Lemma 4.6 of [**V97**]) yields the upper bound

$$\sup_{\beta\in\mathfrak{F}(\alpha)} |g(\Lambda_{13})| \leq \sup_{\gamma\in\mathfrak{k}} |g(\gamma)| \ll PQ^{-1/12}. \tag{5.14}$$

Of course, one has also the trivial upper bound

$$\sup_{\beta\in[0,1)} |g(\Lambda_{13})| \leq P.$$

We therefore deduce from (5.12) that

$$\int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13})h(\Lambda_j)^6|\,d\beta \ll P^{1/6}U^{-1/72}\sup_{\lambda\in\mathbb{R}}\mathcal{U}(\lambda), \tag{5.15}$$

where $U=Q$ when $\alpha\in\mathfrak{K}$, and otherwise $U=1$.

Next, on considering the underlying Diophantine equations, it follows from Theorem 2 of Vaughan [**V86**] that for each $\lambda\in\mathbb{R}$, one has the upper bound

$$\int_0^1 |h(b_j\gamma+\lambda)|^8\,d\gamma \ll P^5.$$

Meanwhile, Lemma 9 of [**BW06**] yields the estimate

$$\sup_{\lambda\in\mathbb{R}}\int_{\mathfrak{M}_0} |g(b_{13}\gamma)|^{5/2}|h(b_j\gamma+\lambda)|^2\,d\gamma \ll P^{3/2}.$$

By applying Hölder's inequality to the integral on the right hand side of (5.13), therefore, we obtain

$$\begin{aligned}\mathcal{U}(\lambda) &\le \Big(\int_0^1 |h(b_j\gamma+\lambda)|^8\,d\gamma\Big)^{2/3}\Big(\int_{\mathfrak{M}_0} |g(b_{13}\gamma)|^{5/2}|h(b_j\gamma+\lambda)|^2\,d\gamma\Big)^{1/3}\\ &\ll (P^5)^{2/3}(P^{3/2})^{1/3}.\end{aligned}$$

On substituting the latter estimate into (5.15), we may conclude that

$$\int_{\mathfrak{F}(\alpha)} |g(\Lambda_{13})h(\Lambda_j)^6|\,d\beta \ll P^4U^{-1/72},$$

with U defined as in the sequel to (5.15). The estimates (5.10) and (5.11) that we seek to establish are then immediate, and in view of our earlier discussion this suffices already to complete the proof of the lemma. □

We now employ the bounds supplied by Lemma 8 to prune the integral on the right hand side of (5.6), making use also of an argument similar to that used in the proof of this lemma. Applying these estimates within the aforementioned equation, we obtain the bound

$$\mathcal{J}(\mathfrak{E}) \ll P^{s-9}\mathcal{K}(\mathfrak{k}\cap\mathfrak{M}) + P^{s-9}Q^{-1/72}\mathcal{K}(\mathfrak{K}), \tag{5.16}$$

where we write

$$\mathcal{K}(\mathfrak{B}) = \int_{\mathfrak{B}} |g(a_1\alpha)H_1(\alpha)|\,d\alpha. \tag{5.17}$$

But in view of (5.4), when $\mathfrak{B}\subseteq\mathfrak{M}$, an application of Hölder's inequality to (5.17) yields

$$\mathcal{K}(\mathfrak{B}) \le \prod_{j=8}^{12}\bigg(\Big(\sup_{\alpha\in\mathfrak{B}}|g(a_1\alpha)|\Big)^{1/58}\mathcal{L}_{1,j}^{14/29}\mathcal{L}_{2,j}^{15/29}\bigg)^{1/5}, \tag{5.18}$$

where for $8\le j\le 12$ we put

$$\mathcal{L}_{1,j} = \int_{\mathfrak{M}} |g(a_1\alpha)|^{57/28}|h(a_j\alpha)|^2\,d\alpha$$

and

$$\mathcal{L}_{2,j} = \int_0^1 |h(a_j\alpha)|^{39/5}\,d\alpha.$$

The integral $\mathcal{L}_{1,j}$ may be estimated by applying Lemma 9 of [**BW06**], and $\mathcal{L}_{2,j}$ via Theorem 2 of Brüdern and Wooley [**BW01**]. Thus we have

$$\mathcal{L}_{1,j} \ll P^{29/28} \quad \text{and} \quad \mathcal{L}_{2,j} \ll P^{24/5} \quad (8 \le j \le 12). \tag{5.19}$$

But as in the argument leading to the estimate (5.14) in the proof of Lemma 8, one has also

$$\sup_{\alpha \in \mathfrak{k}} |g(a_1\alpha)| \ll PQ^{-1/12}. \tag{5.20}$$

Thus, on making use in addition of the trivial estimate $|g(a_1\alpha)| \le P$ valid uniformly in α, and substituting this and the estimates (5.19) and (5.20) into (5.18), we conclude that

$$\mathcal{K}(\mathfrak{k} \cap \mathfrak{M}) \ll P^3 Q^{-1/696} \quad \text{and} \quad \mathcal{K}(\mathfrak{K}) \ll P^3.$$

In this way, we deduce from (5.16) that $\mathcal{J}(\mathfrak{C}) \ll P^{s-6}Q^{-1/696}$. The estimate $\mathcal{J}(\mathfrak{n}) \ll P^{s-6}Q^{-1/696}$ is now confirmed by (5.2), so that by the discussion following that equation, we arrive at the desired lower bound $\mathcal{N}_s(P) \gg P^{s-6}$ for the systems (1.1) of type III under consideration. This completes the proof of Theorem 2 for the latter systems, and so we may turn our attention in the next section to systems of types I and II.

6. An exceptional approach to systems of types I and II

Systems of type II split into two almost separate diagonal cubic equations linked by a single variable. Here we may apply the main ideas from our recent collaboration with Kawada [**BKW01a**] in order to show that this linked cubic variable is almost always simultaneously as often as expected equal both to the first and to the second residual diagonal cubic. A lower bound for $\mathcal{N}_s(P)$ of the desired strength follows with ease. Although systems of type I are accessible in a straightforward fashion to the modern theory of cubic smooth Weyl sums (see, for example, [**V89**] and [**W00**]), we are able to avoid detailed discussion by appealing to the main result underpinning the analysis of type II systems.

In preparation for the statement of the basic estimate of this section, we require some notation. When t is a natural number, and $c_1, \dots, c_t$ are natural numbers, let $R_t(m; \mathbf{c})$ denote the number of positive integral solutions of the equation

$$c_1x_1^3 + c_2x_2^3 + \dots + c_tx_t^3 = m. \tag{6.1}$$

In addition, let η be a positive number with $(c_1 + c_2)\eta < 1/4$ sufficiently small in the context of the estimate (2.2), and put $\nu = 16(c_1 + c_2)\eta$. Finally, recall from (5.3) that $\tau = (1/4 - \xi)/3 > 10^{-4}$.

Theorem 9. *Suppose that t is a natural number with $t \ge 6$, and let $c_1, \dots, c_t$ be natural numbers satisfying $(c_1, \dots, c_t) = 1$. Then for each natural number d there is a positive number Δ, depending at most on $\mathbf{c}$ and d, with the property that the set $\mathcal{E}_t(P)$, defined by*

$$\mathcal{E}_t(P) = \{n \in \mathbb{N} : \nu P d^{-1/3} < n \le P d^{-1/3} \text{ and } R_t(dn^3; \mathbf{c}) < \Delta P^{t-3}\},$$

has at most $P^{1-\tau}$ elements.

We note that the conclusion of the theorem for $t \ge 7$ is essentially classical, and indeed one may establish that $\text{card}(\mathcal{E}_t(P)) = O(1)$ under the latter hypothesis. It is, however, painless to add these additional cases to the primary case $t = 6$, and this permits economies later in this section. Much improvement is possible in the estimate for card($\mathcal{E}_t(P)$) even when $t = 6$ (see Brüdern, Kawada and Wooley [**BKW01a**] for the ideas necessary to save a relatively large power of P). Here we briefly sketch a proof of Theorem 9 that employs a straightforward approach to the problem.

PROOF. Let $\mathfrak{B} \subseteq [0, 1)$ be a measurable set, and consider a natural number m. If we define the Fourier coefficient $\Upsilon_t(m; \mathfrak{B})$ by

$$\Upsilon_t(m;\mathfrak{B}) = \int_{\mathfrak{B}} g(c_1\alpha)g(c_2\alpha)h(c_3\alpha)h(c_4\alpha)\dots h(c_t\alpha)e(-m\alpha)\,d\alpha, \tag{6.2}$$

then it follows from orthogonality that for each $m \in \mathbb{N}$, one has

$$\Upsilon_t(m;[0,1)) \le R_t(m;\mathbf{c}). \tag{6.3}$$

Recall the definition of the sets of major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ from §5. We observe that the methods of Wooley [**W00**] apply to provide the mean value estimate

$$\int_0^1 |g(c_i\alpha)^2 h(c_j\alpha)^4|\,d\alpha \ll P^{3+\xi+\varepsilon} \quad (i = 1, 2 \text{ and } 3 \le j \le t). \tag{6.4}$$

In addition, whenever u is a real number with $u \ge 7.7$, it follows from Theorem 2 of Brüdern and Wooley [**BW01**] that

$$\int_0^1 |h(c_j\alpha)|^u\,d\alpha \ll P^{u-3} \quad (3 \le j \le t). \tag{6.5}$$

Finally, we define the singular series

$$\mathfrak{S}_t(m) = \sum_{q=1}^{\infty} q^{-t} \sum_{\substack{a=1\\(a,q)=1}}^{q} S_1(q,a)S_2(q,a)\dots S_t(q,a)e(-ma/q),$$

where we write

$$S_i(q,a) = \sum_{r=1}^{q} e(c_i a r^3/q) \quad (1 \le i \le t).$$

Then in view of (6.5), the presence of two classical Weyl sums within the integral on the right hand side of (6.2) permits the use of the argument applied by Vaughan in §5 of [**V89**] so as to establish that when τ is a positive number sufficiently small in terms of η, one has

$$\Upsilon_t(m;\mathfrak{M}) = \mathcal{C}_t(\eta;m)\mathfrak{S}_t(m)m^{t/3-1} + O(P^{t-3}(\log P)^{-\tau}),$$

where $\mathcal{C}_t(\eta; m)$ is a non-negative number related to the singular integral. When $\nu^3P^3 < m \le P^3$, it follows from Lemma 8.5 of [**W91**] (see also Lemma 5.4 of [**V89**]) that $\mathcal{C}_t(\eta; m) \gg 1$, in which the implicit constant depends at most on t, $\mathbf{c}$ and η. The methods of Chapter 4 of [**V97**] (see, in particular, Theorem 4.5) show that $\mathfrak{S}_t(m) \gg 1$ uniformly in m, with an implicit constant depending at most on t and $\mathbf{c}$. Here it may be worth remarking that a homogenised version of the representation problem (6.1) defines a diagonal cubic equation in $t+1 \ge 7$ variables. Non-singular p-adic solutions of the latter equation are guaranteed by the work of Lewis [**L57**], and the coprimality of the coefficients $c_1, c_2, \dots, c_t$ ensures that a p-adic solution of the homogenised equation may be found in which the homogenising variable is equal to 1. Thus the existence of non-singular p-adic solutions for the equation (6.1) is assured, and it is this observation that permits us to conclude that $\mathfrak{S}_t(m) \gg 1$.

Our discussion thus far permits us to conclude that when Δ is a positive number sufficiently small in terms of t, $\mathbf{c}$ and η, then for each $m \in (\nu^3 P^3, P^3]$ one has $\Upsilon_t(m; \mathfrak{M}) > 2\Delta P^{t-3}$. But $\Upsilon_t(m; [0,1)) = \Upsilon_t(m; \mathfrak{M}) + \Upsilon_t(m; \mathfrak{m})$, and so it follows from (6.2) and (6.3) that for each $n \in \mathcal{E}_t(P)$, one has

$$|\Upsilon_t(dn^3; \mathfrak{m})| > \Delta P^{t-3}. \tag{6.6}$$

When $n \in \mathcal{E}_t(P)$, we now define σ_n via the relation $|\Upsilon_t(dn^3; \mathfrak{m})| = \sigma_n \Upsilon_t(dn^3; \mathfrak{m})$, and then put

$$K_t(\alpha) = \sum_{n \in \mathcal{E}_t(P)} \sigma_n e(-dn^3 \alpha).$$

Here, in the event that $\Upsilon_t(dn^3; \mathfrak{m}) = 0$, we put $\sigma_n = 0$. Consequently, on abbreviating $\mathrm{card}(\mathcal{E}_t(P))$ to E_t, we find that by summing the relation (6.6) over $n \in \mathcal{E}_t(P)$, one obtains

$$E_t \Delta P^{t-3} < \int_{\mathfrak{m}} g(c_1\alpha) g(c_2\alpha) h(c_3\alpha) h(c_4\alpha) \dots h(c_t\alpha) K_t(\alpha)\, d\alpha. \tag{6.7}$$

An application of Lemma 6 within (6.7) reveals that

$$E_t \Delta P^{t-3} \ll \max_{i=1,2} \max_{3 \le j \le t} \int_{\mathfrak{m}} |g(c_i\alpha)^2 h(c_j\alpha)^{t-2} K_t(\alpha)|\, d\alpha.$$

On making a trivial estimate for $h(c_j\alpha)$ in case $t > 6$, we find by applying Schwarz's inequality that there are indices $i \in \{1, 2\}$ and $j \in \{3, 4, \dots, t\}$ for which

$$E_t \Delta P^{t-3} \ll \Big(\sup_{\alpha \in \mathfrak{m}} |g(c_i\alpha)| \Big) P^{t-6} \mathcal{T}_1^{1/2} \mathcal{T}_2^{1/2},$$

where we write

$$\mathcal{T}_1 = \int_0^1 |g(c_i\alpha)^2 h(c_j\alpha)^4|\, d\alpha \quad \text{and} \quad \mathcal{T}_2 = \int_0^1 |h(c_j\alpha)^4 K_t(\alpha)^2|\, d\alpha.$$

The first of the latter integrals can plainly be estimated via (6.4), and a consideration of the underlying Diophantine equation reveals that the second may be estimated in similar fashion. Thus, on making use of the enhanced version of Weyl's inequality (Lemma 1 of [**V86**]) by now familiar to the reader, we arrive at the estimate

$$E_t \Delta P^{t-3} \ll (P^{3/4+\varepsilon})(P^{t-6})(P^{3+\xi+\varepsilon}) \ll P^{t-2-2\tau+2\varepsilon}.$$

The upper bound $E_t \le P^{1-\tau}$ now follows whenever P is sufficiently large in terms of t, $\mathbf{c}$, η, Δ and τ. This completes the proof of the theorem. $\square$

We may now complete the proof of Theorem 2 for systems of type II. From the discussion in §3, we may suppose that $s \ge 13$, that $7 \le q_0 \le s - 6$, and that amongst the forms Λ_i $(1 \le i \le s)$ there are precisely 3 equivalence classes, one of which has multiplicity 1. By taking suitable linear combinations of the equations (1.1), and by relabelling the variables if necessary, it thus suffices to consider the pair of equations

$$\begin{aligned} a_1 x_1^3 + \dots + a_r x_r^3 \qquad\qquad\qquad\qquad & = d_1 x_s^3, \\ b_{r+1} x_{r+1}^3 + \dots + b_{s-1} x_{s-1}^3 & = d_2 x_s^3, \end{aligned} \tag{6.8}$$

where we have written $d_1 = -a_s$ and $d_2 = -b_s$, both of which we may suppose to be non-zero. We may apply the substitution $x_j \to -x_j$ whenever necessary so as to ensure that all of the coefficients in the system (6.8) are positive. Next write A and B for the greatest common divisors of $a_1, \ldots, a_r$ and $b_{r+1}, \ldots, b_{s-1}$ respectively. On replacing x_s by ABy, with a new variable y, we may cancel a factor A from the coefficients of the first equation, and likewise B from the second. There is consequently no loss in assuming that $A = B = 1$ for the system (6.8).

In view of the discussion of §3, the hypotheses $s \geq 13$ and $7 \leq q_0 \leq s-6$ permit us to assume that in the system (6.8), one has $r \geq 6$ and $s - r \geq 7$. Let Δ be a positive number sufficiently small in terms of a_i $(1 \leq i \leq r)$, b_j $(r+1 \leq j \leq s-1)$, and d_1, d_2. Also, put $d = \min\{d_1, d_2\}$, $D = \max\{d_1, d_2\}$, and recall that $\nu = 16(c_1+c_2)\eta$. Note here that by taking η sufficiently small in terms of $\mathbf{d}$, we may suppose without loss that $\nu d^{-1/3} < \frac{1}{2}D^{-1/3}$. Then as a consequence of Theorem 9, for all but at most $P^{1-\tau}$ of the integers x_s with $\nu P d^{-1/3} < x_s \leq P D^{-1/3}$ one has $R_r(d_1 x_s^3; \mathbf{a}) \geq \Delta P^{r-3}$, and likewise for all but at most $P^{1-\tau}$ of the same integers x_s one has $R_{s-r-1}(d_2 x_s^3; \mathbf{b}) \geq \Delta P^{s-r-4}$. Thus we see that

$$\begin{aligned} \mathcal{N}_s(P) &\geq \sum_{1 \leq x_s \leq P} R_r(d_1 x_s^3; \mathbf{a}) R_{s-r-1}(d_2 x_s^3; \mathbf{b}) \\ &\gg (P - 2P^{1-\tau})(P^{r-3})(P^{s-r-4}). \end{aligned}$$

The bound $\mathcal{N}_s(P) \gg P^{s-6}$ that we sought in order to confirm Theorem 2 for type II systems is now apparent.

The only remaining situations to consider concern type I systems with $s \geq 13$ and $7 \leq q_0 \leq s-6$. Here the simultaneous equations take the shape

$$\begin{aligned} a_1 x_1^3 + \cdots + a_{r-1} x_{r-1}^3 &= d_1 x_r^3, \\ b_{r+1} x_{r+1}^3 + \cdots + b_{s-1} x_{s-1}^3 &= d_2 x_s^3, \end{aligned} \tag{6.9}$$

with $r \geq 7$ and $s - r \geq 7$. As in the discussion of type II systems, one may make changes of variable so as to ensure that $(a_1, \ldots, a_{r-1}) = 1$ and $(b_{r+1}, \ldots, b_{s-1}) = 1$, and in addition that all of the coefficients in the system (6.9) are positive. But as a direct consequence of Theorem 9, in a manner similar to that described in the previous paragraph, one obtains

$$\begin{aligned} \mathcal{N}_s(P) &\geq \sum_{1 \leq x_r \leq P} \sum_{1 \leq x_s \leq P} R_{r-1}(d_1 x_r^3; \mathbf{a}) R_{s-r-1}(d_2 x_s^3; \mathbf{b}) \\ &\gg (P - P^{1-\tau})^2 (P^{r-4})(P^{s-r-4}) \gg P^{s-6}. \end{aligned}$$

This confirms the lower bound $\mathcal{N}_s(P) \gg P^{s-6}$ for type I systems, and thus the proof of Theorem 2 is complete in all cases.

7. Asymptotic lower bounds for systems of smaller dimension

Although our methods are certainly not applicable to general systems of the shape (1.1) containing 12 or fewer variables, we are nonetheless able to generalise the approach described in the previous section so as to handle systems containing at most 3 distinct coefficient ratios. We sketch below the ideas required to establish such conclusions, leaving the reader to verify the details as time permits. It is appropriate in future investigations of pairs of cubic equations, therefore, to restrict attention to systems containing four or more coefficient ratios.

Theorem 10. *Suppose that $s \geq 11$, and that $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ $(1 \leq j \leq s)$ satisfy the condition that the system* (1.1) *admits a non-trivial solution in $\mathbb{Q}_p$ for* *every prime number p. Suppose in addition that the number of equivalence classes amongst the forms $\Lambda_j = a_j\alpha + b_j\beta$ $(1 \le j \le s)$ is at most 3. Then whenever $q_0 \ge 7$, one has $\mathcal{N}_s(P) \gg P^{s-6}$.*

We note that the hypothesis $q_0 \ge 7$ by itself ensures that there must be at least 3 equivalence classes amongst the forms Λ_j $(1 \le j \le s)$ when $8 \le s \le 12$, and at least 4 equivalence classes when $8 \le s \le 10$. The discussion in the introduction, moreover, explains why it is that the hypothesis $q_0 \ge 7$ must be imposed, at least until such time as the current state of knowledge concerning the density of rational solutions to (single) diagonal cubic equations in six or fewer variables dramatically improves. The class of simultaneous diagonal cubic equations addressed by Theorem 10 is therefore as broad as it is possible to address given the restriction that there be at most three distinct equivalence classes amongst the forms Λ_j $(1 \le j \le s)$. In addition, we note that although, when $s \le 12$, one may have p-adic obstructions to the solubility of the system (1.1) for any prime number p with $p \equiv 1 \pmod 3$, for each fixed system with $s \ge 4$ and $q_0 \ge 3$ such an obstruction must come from at worst a finite set of primes determined by the coefficients $\mathbf{a}, \mathbf{b}$.

We now sketch the proof of Theorem 10. When $s \ge 13$, of course, the desired conclusion follows already from that of Theorem 2. We suppose henceforth, therefore, that s is equal to either 11 or 12. Next, in view of the discussion of §3, we may take suitable linear combinations of the equations and relabel variables so as to transform the system (1.1) to the shape

$$\sum_{i=1}^{l} \lambda_i x_i^3 = \sum_{j=1}^{m} \mu_j y_j^3 = \sum_{k=1}^{n} \nu_k z_k^3, \tag{7.1}$$

with $\lambda_i, \mu_j, \nu_k \in \mathbb{Z} \setminus \{0\}$ $(1 \le i \le l,\ 1 \le j \le m,\ 1 \le k \le n)$, wherein

$$l \ge m \ge n, \quad l + m + n = s, \quad l + n \ge 7 \quad \text{and} \quad m + n \ge 7. \tag{7.2}$$

By applying the substitution $x_i \to -x_i$, $y_j \to -y_j$ and $z_k \to -z_k$ wherever necessary, moreover, it is apparent that we may assume without loss that all of the coefficients in the system (7.1) are positive. In this way we conclude that

$$\mathcal{N}_s(P) \ge \sum_{1 \le N \le P^3} R_l(N; \boldsymbol{\lambda}) R_m(N; \boldsymbol{\mu}) R_n(N; \boldsymbol{\nu}). \tag{7.3}$$

Finally, we note that the only possible triples (l, m, n) permitted by the constraints (7.2) are $(5, 5, 2)$, $(5, 4, 3)$ and $(4, 4, 4)$ when $s = 12$, and $(4, 4, 3)$ when $s = 11$. We consider these four triples (l, m, n) in turn. Throughout, we write τ for a sufficiently small positive number.

We consider first the triple of multiplicities $(5, 5, 2)$. Let $(\nu_1, \nu_2) \in \mathbb{N}^2$, and denote by $\mathfrak{X}$ the multiset of integers $\{\nu_1 z_1^3 + \nu_2 z_2^3 : z_1, z_2 \in \mathcal{A}(P, P^\eta)\}$. Consider a 5-tuple $\boldsymbol{\xi}$ of natural numbers, and denote by $\mathfrak{X}(P; \boldsymbol{\xi})$ the multiset of integers $N \in \mathfrak{X} \cap [\frac{1}{2}P^3, P^3]$ for which the equation $\xi_1 u_1^3 + \cdots + \xi_5 u_5^3 = N$ possesses a p-adic solution $\mathbf{u}$ for each prime p. It follows from the hypotheses of the statement of the theorem that the multiset $\mathfrak{X}(P; \boldsymbol{\lambda}; \boldsymbol{\mu}) = \mathfrak{X}(P; \boldsymbol{\lambda}) \cap \mathfrak{X}(P; \boldsymbol{\mu})$ is non-empty. Indeed, by considering a suitable arithmetic progression determined only by $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, a simple counting argument establishes that $\mathrm{card}(\mathfrak{X}(P; \boldsymbol{\lambda}; \boldsymbol{\mu})) \gg P^2$. Then by the methods of [**BKW01a**] (see also the discussion following the statement of Theorem 1.2 of [**BKW01b**]), one has the lower bound $R_5(N; \boldsymbol{\lambda}) \gg P^2$ for each $N \in \mathfrak{X}(P;\boldsymbol{\lambda};\boldsymbol{\mu})$ with at most $O(P^{2-\tau})$ possible exceptions. Similarly, one has $R_5(N;\boldsymbol{\mu}) \gg P^2$ for each $N \in \mathfrak{X}(P;\boldsymbol{\lambda};\boldsymbol{\mu})$ with at most $O(P^{2-\tau})$ possible exceptions. Thus we see that for systems with coefficient ratio multiplicity profile $(5,5,2)$, one has the lower bound

$$\begin{aligned} \mathcal{N}_{12}(P) &\geq \sum_{N\in\mathfrak{X}(P;\boldsymbol{\lambda};\boldsymbol{\mu})} R_5(N;\boldsymbol{\lambda})R_5(N;\boldsymbol{\mu}) \\ &\gg (P^2-2P^{2-\tau})(P^2)^2 \gg P^6. \end{aligned} \tag{7.4}$$

Consider next the triple of multiplicities $(5,4,3)$. Let $(\nu_1,\nu_2,\nu_3) \in \mathbb{N}^3$, and take $\tau > 0$ as before. We now denote by $\mathfrak{Y}$ the multiset of integers

$$\{\nu_1 z_1^3 + \nu_2 z_2^3 + \nu_3 z_3^3 \,:\, z_1, z_2, z_3 \in \mathcal{A}(P,P^\eta)\}.$$

Consider a v-tuple $\boldsymbol{\xi}$ of natural numbers with $v \geq 4$, and denote by $\mathfrak{Y}_v(P;\boldsymbol{\xi})$ the multiset of integers $N \in \mathfrak{Y} \cap [\frac{1}{2}P^3, P^3]$ for which the equation $\xi_1 u_1^3 + \cdots + \xi_v u_v^3 = N$ possesses a p-adic solution $\mathbf{u}$ for each prime p. The hypotheses of the statement of the theorem ensure that the multiset $\mathfrak{Y}(P;\boldsymbol{\lambda};\boldsymbol{\mu}) = \mathfrak{Y}_5(P;\boldsymbol{\lambda}) \cap \mathfrak{Y}_4(P;\boldsymbol{\mu})$ is non-empty. Indeed, again by considering a suitable arithmetic progression determined only by $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$, one may show that $\mathrm{card}(\mathfrak{Y}(P;\boldsymbol{\lambda};\boldsymbol{\mu})) \gg P^3$. When $s \geq 4$, the methods of [**BKW01a**] may on this occasion be applied to establish the lower bound $R_5(N;\boldsymbol{\lambda}) \gg P^2$ for each $N \in \mathfrak{Y}(P;\boldsymbol{\lambda};\boldsymbol{\mu})$, with at most $O(P^{3-\tau})$ possible exceptions. Likewise, one obtains the lower bound $R_4(N;\boldsymbol{\mu}) \gg P$ for each $N \in \mathfrak{Y}(P;\boldsymbol{\lambda};\boldsymbol{\mu})$, with at most $O(P^{3-\tau})$ possible exceptions. Thus we find that for systems with coefficient ratio multiplicity profile $(5,4,3)$, one has the lower bound

$$\begin{aligned} \mathcal{N}_{12}(P) &\geq \sum_{N\in\mathfrak{Y}(P;\boldsymbol{\lambda};\boldsymbol{\mu})} R_5(N;\boldsymbol{\lambda})R_4(N;\boldsymbol{\mu}) \\ &\gg (P^3-2P^{3-\tau})(P^2)(P) \gg P^6. \end{aligned} \tag{7.5}$$

The triple of multiplicities $(4,4,3)$ may plainly be analysed in essentially the same manner, so that

$$\begin{aligned} \mathcal{N}_{11}(P) &\geq \sum_{N\in\mathfrak{Y}(P;\boldsymbol{\lambda};\boldsymbol{\mu})} R_4(N;\boldsymbol{\lambda})R_4(N;\boldsymbol{\mu}) \\ &\gg (P^3-2P^{3-\tau})(P)^2 \gg P^5. \end{aligned} \tag{7.6}$$

An inspection of the cases listed in the aftermath of equation (7.3) reveals that it is only the multiplicity triple $(4,4,4)$ that remains to be tackled. But here conventional exceptional set technology in combination with available estimates for cubic Weyl sums may be applied. Consider a 4-tuple $\boldsymbol{\xi}$ of natural numbers, and denote by $\mathfrak{Z}(P;\boldsymbol{\xi})$ the set of integers $N \in [\frac{1}{2}P^3, P^3]$ for which the equation $\xi_1 u_1^3 + \cdots + \xi_4 u_4^3 = N$ possesses a p-adic solution $\mathbf{u}$ for each prime p. It follows from the hypotheses of the statement of the theorem that the set

$$\mathfrak{Z}(P;\boldsymbol{\lambda};\boldsymbol{\mu};\boldsymbol{\nu}) = \mathfrak{Z}(P;\boldsymbol{\lambda}) \cap \mathfrak{Z}(P;\boldsymbol{\mu}) \cap \mathfrak{Z}(P;\boldsymbol{\nu})$$

is non-empty. But the estimates of Vaughan [**V86**] permit one to prove that the lower bound $R_4(N;\boldsymbol{\lambda}) \gg P$ holds for each $N \in \mathfrak{Z}(P;\boldsymbol{\lambda};\boldsymbol{\mu};\boldsymbol{\nu})$ with at most $O(P^3(\log P)^{-\tau})$ possible exceptions, and likewise when $R_4(N;\boldsymbol{\lambda})$ is replaced by $R_4(N;\boldsymbol{\mu})$ or $R_4(N;\boldsymbol{\nu})$. Thus, for systems with coefficient ratio multiplicity profile

$(4,4,4)$, one arrives at the lower bound

$$\begin{aligned}\mathcal{N}_{12}(P) &\geq \sum_{N\in\mathfrak{Z}(\boldsymbol{\lambda};\boldsymbol{\mu};\boldsymbol{\nu})} R_4(N;\boldsymbol{\lambda})R_4(N;\boldsymbol{\mu})R_4(N;\boldsymbol{\nu}) \\ &\gg (P^3-3P^3(\log P)^{-\tau})(P)^3 \gg P^6.\end{aligned} \tag{7.7}$$

On collecting together (7.4), (7.5), (7.6) and (7.7), the proof of the theorem is complete.

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Institut für Algebra und Zahlentheorie, Pfaffenwaldring 57, Universität Stuttgart, D-70511 Stuttgart, Germany

E-mail address: `bruedern@mathematik.uni-stuttgart.de`

Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, U.S.A.

E-mail address: `wooley@umich.edu`