Additive representation in thin sequences, VII: restricted moments of the number of representations.

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1. Introduction.

Amongst our opera devoted to additive problems restricted to thin polynomial sequences (see in particular [3], [4], [5]), the tertiary part is devoted to estimates for exceptional sets associated with the expected asymptotic formula for the number of representations of prescribed type. While such estimates lead directly to lower bounds of the anticipated size for likewise restricted moments of the number of representations, uncertainties concerning integers associated with the exceptional set prohibit any immediate inference of asymptotic formulae for such moments. The purpose of this paper is to develop methods that establish such asymptotic formulae, thereby avoiding the aforementioned uncertainties.

Continuing the tradition of our previous excursions in this series, we illustrate our ideas with a discussion of Waring's problem for cubes. Denote by $R_s(n)$ the number of representations of n as the sum of s cubes of positive integers. A heuristic application of the circle method suggests that for $s \geq 4$, one should have the asymptotic formula

$$R_s(n) = \Gamma(4/3)^s \Gamma(s/3)^{-1} \mathfrak{S}_s(n) n^{s/3-1} + o(n^{s/3-1}), \tag{1.1}$$

as $n \to \infty$, where

$$\mathfrak{S}_s(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \big(q^{-1} S_3(q,a)\big)^s e(-na/q),$$

in which we write

$$S_3(q,a) = \sum_{r=1}^{q} e(ar^3/q),$$

and $e(z) = \exp(2\pi i z)$. It is useful to recall that when $s \geq 4$, the *singular series* $\mathfrak{S}_s(n)$ is known to satisfy the lower bound $\mathfrak{S}_s(n) \gg 1$ (see Theorem 4.5 of Vaughan [13]), whence the relation (1.1) constitutes an honest asymptotic formula.

The validity of the expected asymptotic formula (1.1) would imply corresponding formulae for the moments of $R_s(n)$. In particular, one

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expects that for $s \geq 4$ and for all positive values of h, one should have the asymptotic formulae

$$\sum_{n \leq x} R_s(n)^h = \mathcal{C}_1(s,h) x^{h(s/3-1)+1} + o(x^{h(s/3-1)+1}), \tag{1.2}$$

as $x \to \infty$, where the quantity $\mathcal{C}_1(s,h)$ is defined by the relation

$$\mathcal{C}_1(s,h) = \frac{1}{h(s/3-1)+1} \left(\frac{\Gamma(4/3)^s}{\Gamma(s/3)} \right)^h \mathfrak{C}_1(s,h),$$

in which

$$\mathfrak{C}_1(s,h) = \lim_{y \to \infty} y^{-1} \sum_{n \leq y} \mathfrak{S}_s(n)^h.$$

We note that the latter limit does indeed exist, as we establish in Lemma 1 below for $s \geq 5$, the corresponding result for $s = 4$ following with a little additional effort. When $h = 2$, this desired conclusion is known to hold in all cases, the most difficult situation with $s = 4$ following from the celebrated work of Vaughan [11] (see Theorem 3 of the latter paper). For larger values of h, such asymptotic formulae have entered the literature only very recently with work of Brüdern and Wooley [6]. Here the range of s for which the formula (1.2) holds becomes more restricted as h increases. Thus, for example, when $h = 3$ the formula (1.2) is known to hold only for $s \geq 5$ (see Theorems 1.1 and 1.2 of [6]).

We now turn to analogues of the formula (1.2) in which the summation on the left hand side of the relation is restricted to values of a thin polynomial sequence. It is convenient henceforth to describe a polynomial $\phi \in \mathbb{Q}[t]$ as being an *integral polynomial* if, whenever the parameter t is an integer, then $\phi(t)$ is also an integer. In such circumstances, we write a_ϕ for the leading coefficient of $\phi(t)$, and we write d_ϕ for the degree of ϕ. When $a_\phi > 0$, the conjectured analogue of the asymptotic formula (1.2) now becomes

$$\sum_{n \leq x} R_s(\phi(n))^h = \mathcal{C}_\phi(s,h) x^{d_\phi h(s/3-1)+1} + o(x^{d_\phi h(s/3-1)+1}), \tag{1.3}$$

where the quantity $\mathcal{C}_\phi(s,h)$ is defined by the relation

$$\mathcal{C}_\phi(s,h) = \frac{a_\phi^{h(s/3-1)}}{d_\phi h(s/3-1)+1} \left(\frac{\Gamma(4/3)^s}{\Gamma(s/3)} \right)^h \mathfrak{C}_\phi(s,h),$$

in which

$$\mathfrak{C}_\phi(s,h) = \lim_{y \to \infty} y^{-1} \sum_{n \leq y} \mathfrak{S}_s(\phi(n))^h. \tag{1.4}$$

Again, the existence of this limit is assured by Lemma 1 below for $s \geq 5$, and may be established for $s = 4$ with greater effort. We remark also that the formula (1.4) may be replaced for integral h by one more reminiscent of the definition of a conventional singular series. Thus, for example, one may show that

$$\mathfrak{C}_\phi(s,2) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,b,q)=1}}^{q} \sum_{b=1}^{q} q^{-2s-1} S_3(q,a)^s S_3(q,-b)^s S_\phi(q,b-a),$$

where

$$S_\phi(q,t) = (d_\phi!)^{-1} \sum_{r=1}^{d_\phi! q} e(t\phi(r)/q).$$

Our first conclusion provides asymptotic formulae of the type (1.3) when $\phi(t)$ is a quadratic polynomial. We refer the reader to the proof of this result in §3 for more precise but technical estimates corresponding to minor arc behaviour alone.

Theorem 1. *Suppose that $\phi(n)$ is an integral quadratic polynomial with positive leading coefficient. Then the anticipated asymptotic formula* (1.3) *holds* (i) *when $s = 6$ and $0 < h \leq 2$, and* (ii) *when $s = 7$ and $0 < h \leq 4$.*

In view of Vaughan's work [11] concerning the asymptotic formula for sums of eight cubes, of course, the formula (1.3) holds for all positive numbers h when $s \geq 8$.

In §3 we also establish a corresponding conclusion for cubic polynomials.

Theorem 2. *Suppose that $\phi(n)$ is an integral cubic polynomial with positive leading coefficient. Then the anticipated asymptotic formula* (1.3) *holds when $s = 7$ and $0 < h \leq 2$.*

Even in this brief excursion on the topic of asymptotic formulae for restricted moments, two further examples deserve our attention. We note first that an analogue of the asymptotic formula (1.3) holds also in the binary Goldbach problem. Let $r(n)$ denote the number of representations of the integer n as the sum of two prime numbers, and let $\phi(t)$ be an integral polynomial with positive leading coefficient. Then it is a simple matter to establish that for each positive number h, one has

$$\sum_{n \leq x} r\big(2\phi(n)\big)^h = \mathcal{B}_\phi(h) \frac{x^{d_\phi h+1}}{(\log x)^{2h}} + o(x^{d_\phi h+1}(\log x)^{-2h}), \tag{1.5}$$

where the quantity $\mathcal{B}_\phi(h)$ is defined by the relation

$$\mathcal{B}_\phi(h)=\frac{\mathfrak{B}_\phi(h)}{d_\phi h+1}\Bigl(\frac{4a_\phi}{d_\phi^2}\Bigr)^h\prod_{p>2}\Bigl(1-\frac{1}{(p-1)^2}\Bigr)^h,$$

in which

$$\mathfrak{B}_\phi(h)=\lim_{y\to\infty}y^{-1}\sum_{n\le y}\prod_{\substack{p|\phi(n)\\ p>2}}\Bigl(\frac{p-1}{p-2}\Bigr)^h.$$

We note that the existence of the latter limit may be readily confirmed by means of a routine argument that need not detain us here. Meanwhile, in short, the formula (1.5) follows on observing that the methods of [2] may be adapted to show that there are at most $O(x/\log x)$ integers n, with $1\le n\le x$, for which one has $\phi(n)>1$ and

$$\Bigl|r\bigl(2\phi(n)\bigr)-C(\phi(n))\frac{2\phi(n)}{(\log 2\phi(n))^2}\Bigr|>\frac{\phi(n)}{(\log\phi(n))^3}.$$

Here, we have written $C(m)$ for the familiar Goldbach constant defined by

$$C(m)=2\prod_{p>2}\Bigl(1-\frac{1}{(p-1)^2}\Bigr)\prod_{\substack{p|m\\ p>2}}\Bigl(\frac{p-1}{p-2}\Bigr).$$

But sieve methods establish that whenever $\phi(n)>1$, one has

$$r\bigl(2\phi(n)\bigr)\ll\phi(n)/(\log\phi(n))^2,$$

whence the latter set of exceptional integers make a negligible contribution to the left hand side of (1.5). Also, the integers n with $\phi(n)\le 1$ are trivially negligible. The desired conclusion is then immediate via partial summation.

Finally, we note that the methods of this paper extend routinely to arbitrary powers. The following conclusion on sums of kth powers, with k large, suffices to illustrate the associated ideas. We restrict attention at this stage to quadratic polynomials for the sake of elegance rather than for reason of technical obstructions. We write $R_{s,k}(n)$ for the number of representations of n as the sum of s positive integral kth powers.

Theorem 3. *Let $\phi(t)$ be an integral quadratic polynomial with positive leading coefficient a_ϕ, and let k be a positive integer with $k\ge 3$. For each positive number h, and every integer s with $s\ge k+2$, define*

$$\mathcal{A}_\phi(s,h)=\frac{a_\phi^{h(s/k-1)}}{2h(s/k-1)+1}\left(\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\right)^h\mathfrak{A}_\phi(s,h),$$

where

$$\mathfrak{A}_\phi(s,h)=\lim_{y\to\infty} y^{-1}\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n))^h,$$

and for natural numbers m we write

$$\mathfrak{S}_{s,k}(m)=\sum_{q=1}^{\infty}\sum_{\substack{a=1\\(a,q)=1}}^{q}\big(q^{-1}S_k(q,a)\big)^s e(-ma/q),$$

in which

$$S_k(q,a)=\sum_{r=1}^{q} e(ar^k/q).$$

Then whenever h is a fixed positive number with $h\ge 2$, the asymptotic formula

$$\sum_{n\le x} R_{s,k}(\phi(n))^h=\mathcal{A}_\phi(s,h)x^{2h(s/k-1)+1}+o(x^{2h(s/k-1)+1}) \tag{1.6}$$

holds for

$$s\ge\Big(1-\frac{1}{2h}\Big)k^2(\log k+\log\log k+O(1)).$$

Once again, we note that the existence of the limit $\mathfrak{A}_\phi(s,h)$ is assured by Lemma 1 below for $s\ge k+2$. Finally, we recall that whenever $s\ge 4k$, one has $\mathfrak{S}_{s,k}(n)\gg 1$ for all integers n (see Theorem 4.6 of Vaughan [13]), so that Theorem 3 provides a proper asymptotic formula (1.6).

We refer the reader to our earlier paper [3] for a lengthy discussion concerning the basic plan of attack on problems associated with exceptional sets restricted to thin polynomial sequences. As in our earlier papers, the key idea is to introduce an exponential sum that encodes information concerning abnormal deviations from the expected asymptotic formula (1.1) within the sequence of integers n under investigation. Mean value estimates involving this exponential sum may then be exploited to good effect, the preservation of underlying arithmetic information representing a critical advantage of our approach over more traditional applications of Bessel's inequality.

Throughout, the letter ε will denote a sufficiently small positive number. We take P to be the basic parameter, a large real number depending at most on ε, k, s, h, and any coefficients and degrees of implicit polynomials if necessary. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, k, s, h and implicit polynomials. Summations start at 1 unless indicated otherwise. In an effort to simplify our analysis, we adopt the following convention concerning the parameter ε. Whenever ε appears in a statement, we assert that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε.

2. Main terms.

No substantial difficulty is involved in computations associated with the main terms of the formulae that we aim to establish, and here we dispose of this routine work in a form commonly applicable to all of our theorems. Thus we begin by adopting the notation employed in the statement of Theorem 3, except that now we suppose that $\phi(t)$ is an integral polynomial of degree d_ϕ with positive leading coefficient a_ϕ.

For a large real number x, we define

$$P = \phi(x)^{1/k} \qquad \text{and} \qquad f(\alpha) = \sum_{1 \le m \le P} e(\alpha m^k), \tag{2.1}$$

so that for $1 \le n \le x$, we have

$$R_{s,k}(\phi(n)) = \int_0^1 f(\alpha)^s e(-\phi(n)\alpha) d\alpha.$$

We then dissect the unit interval $[0, 1)$ in accordance with §4.4 of Vaughan [13]. Let $\mathfrak{M}$ denote the union of the intervals

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) \,:\, |q\alpha - a| \le P^{1-k}/(2k)\,\},$$

with $0 \le a \le q \le P/(2k)$ and $(a, q) = 1$. The contribution of $\mathfrak{M}$ to the last integral is evaluated by Theorem 4.4 of Vaughan [13]. In fact, on writing

$$\mathfrak{m} = [0, 1) \setminus \mathfrak{M} \qquad \text{and} \qquad R_{s,k}(n; \mathfrak{m}) = \int_{\mathfrak{m}} f(\alpha)^s e(-n\alpha)\, d\alpha, \tag{2.2}$$

it follows from the latter theorem that if $s \ge \max\{5, k+1\}$, then there exists a positive real number δ, depending at most on s and k, such that whenever $1 \le \phi(n) \le \phi(x)$, one has

$$\begin{aligned} R_{s,k}(\phi(n)) = \frac{\Gamma(1 + 1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(\phi(n)) \phi(n)^{s/k-1} \\ + R_{s,k}(\phi(n); \mathfrak{m}) + O(\phi(x)^{s/k-1-\delta}). \end{aligned} \tag{2.3}$$

Here, we have written $\mathfrak{S}_{s,k}(m)$ for the familiar singular series defined in the statement of Theorem 3 above.

Our theorems follow from suitable information on the minor arc contribution $R_{s,k}(\phi(n); \mathfrak{m})$, and we make this clear in the form of Lemma 2 below. In this context, when h is a positive number and s is an integer with $s \geq k+2$, we define

$$\mathcal{D}_\phi(s,h) = \frac{a_\phi^{h(s/k-1)}}{d_\phi h(s/k-1)+1}\left(\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\right)^h \mathfrak{D}_\phi(s,h),$$

where

$$\mathfrak{D}_\phi(s,h) = \lim_{y\to\infty} y^{-1}\sum_{n\leq y}\mathfrak{S}_{s,k}(\phi(n))^h. \tag{2.4}$$

In order to confirm the existence of the limit occurring in the definition of the quantity $\mathfrak{D}_\phi(s,h)$, we must digress from the main path leading to Lemma 2 below.

Lemma 1. *With the notation introduced above, when $s \geq k+2$ and h is a positive number, the limit* (2.4) *exists, and whenever N is a large positive number, one has*

$$\left|\mathfrak{D}_\phi(s,h) - N^{-1}\sum_{n\leq N}\mathfrak{S}_{s,k}(\phi(n))^h\right| \leq (\log N)^{-\min\{1,h\}/(4k)}.$$

Moreover, for each positive number θ,

$$\sum_{n\leq N} n^\theta \mathfrak{S}_{s,k}(\phi(n))^h = \frac{N^{\theta+1}}{\theta+1}\mathfrak{D}_\phi(s,h) + O(N^{\theta+1}(\log N)^{-\min\{1,h\}/(4k)}).$$

Proof. We first note that for real numbers h, X and Y satisfying $h>0$, $X\geq 0$ and $X+Y\geq 0$, we have

$$\begin{aligned}(X+Y)^h - X^h &= h\int_X^{X+Y} t^{h-1}\,dt\\ &\leq h|Y|\max\{X^{h-1},\,(X+Y)^{h-1}\},\end{aligned}$$

provided that the latter expression is defined. When $h>1$, we have $(X+Y)^{h-1} \ll X^{h-1}+|Y|^{h-1}$, so

$$(X+Y)^h - X^h \ll |Y|^h + |X^{h-1}Y|. \tag{2.5}$$

When $0<h\leq 1$, we see that if $X>2|Y|>0$, then $X+Y>|Y|$ and

$$(X+Y)^h - X^h \ll |Y|^h, \tag{2.6}$$

while in circumstances wherein $X\leq 2|Y|$ or $Y=0$, the latter inequality holds trivially. We note in addition that when $X<0$ and $X+Y\geq 0$, one has $X+Y\leq Y$ and $|X|<Y$, and thus for $h>0$ the trivial estimate $(X+Y)^h - X^h \ll (X+Y)^h + |X|^h$ again ensures that the estimates (2.5) and (2.6) hold.

Next we recall some basic features of the analysis of singular series in Waring's problem. Define the truncated singular series $\mathfrak{S}_{s,k}(m;Q)$ by

$$\mathfrak{S}_{s,k}(m;Q)=\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\bigl(q^{-1}S_k(q,a)\bigr)^s e(-ma/q).$$

Also, define the multiplicative function $w_k(q)$ by taking

$$w_k(p^{uk+v})=\begin{cases}kp^{-u-1/2}, & \text{when } u\ge 0 \text{ and } v=1,\\ p^{-u-1}, & \text{when } u\ge 0 \text{ and } 2\le v\le k,\end{cases}$$

for prime numbers p, and non-negative integers u and v. Then according to Lemma 3 of Vaughan [12], whenever $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$, one has $q^{-1}S_k(q,a)\ll w_k(q)$, whence

$$\mathfrak{S}_{s,k}(m)-\mathfrak{S}_{s,k}(m;Q)\ll\sum_{q>Q}qw_k(q)^s\ll\sum_{q=1}^{\infty}(q/Q)^{1/(2k)}qw_k(q)^s.$$

In view of our definition of $w_k(q)$, it follows that when u and v are non-negative integers, and p is a prime number, then for each exponent s with $s\ge k+2$, one has

$$(p^{uk+v})^{1+1/(2k)}w_k(p^{uk+v})^s\le\begin{cases}k^sp^{-u-5/4}, & \text{when } u\ge 0 \text{ and } v=1,\\ p^{-u-3/2}, & \text{when } u\ge 0 \text{ and } 2\le v\le k.\end{cases}$$

Thus we see that

$$\begin{aligned}\mathfrak{S}_{s,k}(m)-\mathfrak{S}_{s,k}(m;Q)&\ll Q^{-1/(2k)}\prod_p\left(1+\sum_{l=1}^{\infty}(p^l)^{1+1/(2k)}w_k(p^l)^s\right)\\&\ll Q^{-1/(2k)}\prod_p\bigl(1+4k^sp^{-5/4}\bigr)\ll Q^{-1/(2k)}.\end{aligned}$$

Here the product is implicitly restricted to run over prime numbers p. In particular, since for each integer m one has $0\le\mathfrak{S}_{s,k}(m)\ll 1$, it follows from the estimate (2.5) that when $h>1$, one has

$$\begin{aligned}\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n))^h-\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n);Q)^h&\\ \ll\sum_{n\le y}\bigl(Q^{-1/(2k)}|\mathfrak{S}_{s,k}(\phi(n))|^{h-1}+Q^{-h/(2k)}\bigr)&\\ \ll yQ^{-\min\{1,h\}/(2k)},&\end{aligned}\tag{2.7}$$

whilst for $0<h\le 1$, the concluding estimate follows in the same manner from (2.6).

Observe next that as a function of n, it is apparent that

$$\left(q^{-1}S_k(q,a)\right)^s e(-a\phi(n)/q)$$

is periodic with period dividing $d_\phi!q$. Consequently, when Q is a natural number, the function

$$\mathfrak{S}_{s,k}(\phi(n);Q)=\sum_{q\le Q}\sum_{\substack{a=1\\(a,q)=1}}^{q}\left(q^{-1}S(q,a)\right)^s e(-a\phi(n)/q)$$

is a periodic function of n with period dividing $d_\phi!Q!$. Write

$$\mathfrak{T}_{s,k}(Q)=\frac{1}{d_\phi!Q!}\sum_{n=1}^{d_\phi!Q!}\mathfrak{S}_{s,k}(\phi(n);Q)^h.$$

Then we see that for $y>(d_\phi!Q!)^2$, one has

$$\left|\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n);Q)^h-\frac{y}{d_\phi!Q!}\sum_{n=1}^{d_\phi!Q!}\mathfrak{S}_{s,k}(\phi(n);Q)^h\right|$$
$$\ll\sum_{n=1}^{d_\phi!Q!}|\mathfrak{S}_{s,k}(\phi(n);Q)|^h\ll Q!,$$

whence

$$\left|\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n);Q)^h-y\mathfrak{T}_{s,k}(Q)\right|\ll y^{1/2}.$$

On substituting the latter estimate into (2.7), we find that when Q is a sufficiently large natural number, and $y>(d_\phi!Q!)^2$, then

$$\left|\frac{1}{y}\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n))^h-\mathfrak{T}_{s,k}(Q)\right|\le Q^{-\min\{1,h\}/(3k)}. \tag{2.8}$$

Given a positive number ε, take Q to be a natural number with $Q>(3/\varepsilon)^{3k/\min\{1,h\}}$. Then whenever y_1 and y_2 exceed $(d_\phi!Q!)^2$, we deduce from (2.8) that

$$\left|\frac{1}{y_1}\sum_{n\le y_1}\mathfrak{S}_{s,k}(\phi(n))^h-\frac{1}{y_2}\sum_{n\le y_2}\mathfrak{S}_{s,k}(\phi(n))^h\right|\le 2Q^{-\min\{1,h\}/(3k)}$$
$$<\varepsilon. \tag{2.9}$$

It follows that the sequence

$$\left(\frac{1}{y}\sum_{n\le y}\mathfrak{S}_{s,k}(\phi(n))^h\right)_{y=1}^{\infty}$$

is a Cauchy sequence, and hence has a limit $\mathfrak{D}_\phi(s,h)$, as claimed in the first assertion of the lemma. Moreover, on taking the limit as $y_2 \to \infty$ within (2.9), we find that

$$\left| \frac{1}{y} \sum_{n \le y} \mathfrak{S}_{s,k}(\phi(n))^h - \mathfrak{D}_\phi(s,h) \right| \le 2Q^{-\min\{1,h\}/(3k)}.$$

Since we may take $Q = [(\log y)^{5/6}]$, we conclude that

$$\left| \frac{1}{y} \sum_{n \le y} \mathfrak{S}_{s,k}(\phi(n))^h - \mathfrak{D}_\phi(s,h) \right| \le (\log y)^{-\min\{1,h\}/(4k)},$$

and this completes the proof of the second assertion of the lemma.

The final assertion of the lemma follows at once from the second via partial summation.

We next incorporate the singular series average into a mean value for $R_{s,k}(\phi(n))$ conditional on suitable control of moments of the minor arc contribution.

Lemma 2. *With the notation introduced above, if $s \ge k+2$ and*

$$\sum_{n \le x} |R_{s,k}(\phi(n);\mathfrak{m})|^H = o(x^{d_\phi H(s/k-1)+1}), \tag{2.10}$$

for some positive real number H, then for any h with $0 < h \le H$, one has

$$\sum_{n \le x} R_{s,k}(\phi(n))^h = \mathcal{D}_\phi(s,h) x^{d_\phi h(s/k-1)+1} + o(x^{d_\phi h(s/k-1)+1}).$$

Proof. Let c be the least natural number such that whenever $n \ge c$, one has $\phi(n) \ge 1$. Also, for the sake of concision, write

$$\Xi_h = \sum_{n \le x} |R_{s,k}(\phi(n);\mathfrak{m})|^h.$$

Then, in view of (2.3), we may argue as in (2.5) that when $h > 1$ we have

$$\sum_{c \le n \le x} R_{s,k}(\phi(n))^h - \sum_{c \le n \le x} \Big(\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(\phi(n)) \phi(n)^{s/k-1} \Big)^h \ll \Xi_h + x^{d_\phi(s/k-1)(h-1)} \Xi_1 + x^{d_\phi h(s/k-1-\delta)+1}. \tag{2.11}$$

Now, by Hölder's inequality, we have $\Xi_g \ll x^{(1-g/H)} \Xi_H^{g/H}$ whenever $0 < g \le H$. So, by our assumption (2.10), we see that if $1 < h \le H$, then the right hand side of (2.11) is $o(x^{d_\phi h(s/k-1)+1})$.

On the other hand, recalling the final conclusion of Lemma 1, the second sum on the left hand side of (2.11) is seen to be

$$\mathcal{D}_\phi(s,h)x^{d_\phi h(s/k-1)+1} + o(x^{d_\phi h(s/k-1)+1}).$$

Hence, the desired conclusion follows immediately from (2.11) when $1 < h \le H$, as the contribution of natural numbers $n < c$ is obviously $O(1)$.

When $0 < h \le 1$, we may proceed as above, but use (2.6) in place of (2.5). We then get a formula similar to (2.11) without the term involving Ξ_1 on the right hand side, and we again obtain the desired formula when $0 < h \le \min\{1, H\}$, appealing to (2.10). We thus complete the proof of the lemma.

3. Waring's problem for cubes.

We come to the central part of the paper, and in this section prove Theorems 1 and 2. Therefore, setting $k = 3$, we adopt the notation introduced in the preamble to Lemma 1. Note in particular that now

$$f(\alpha) = \sum_{1\le m\le P} e(\alpha m^3).$$

We first provide useful mean value estimates in certain generality.

Lemma 3. *Let (η_n) be a sequence of complex numbers with $|\eta_n| \le 1$, let $\mathcal{Z}$ be a set of natural numbers, and write Z for the cardinality of $\mathcal{Z}$. Also let ϕ be an integral polynomial with degree at least 2, and define*

$$K(\alpha) = \sum_{n\in\mathcal{Z}} \eta_n e(\alpha\phi(n)).$$

Then, one has

$$\int_0^1 |f(\alpha)K(\alpha)|^2\, d\alpha \ll PZ + P^2(\log P)^2, \tag{3.1}$$

as well as

$$\int_0^1 |f(\alpha)K(\alpha)|^2\, d\alpha \ll PZ + P^{4/3+\varepsilon} + Z^2. \tag{3.2}$$

Proof. The inequality (3.1) is quite similar to (2.15) of [4], but here we need to handle the divisor function a bit more precisely. By orthogonality, the integral estimated in the lemma is bounded above by the number of solutions of the equation

$$m_1^3 - m_2^3 = \phi(n_1) - \phi(n_2), \tag{3.3}$$

with $1 \le m_1,\ m_2 \le P$ and $n_1,\ n_2 \in \mathcal{Z}$. When $m_1 = m_2$, this equation implies either that $n_1 = n_2$ or that n_1 and n_2 are both $O(1)$, whence the number of solutions of this type is $O(PZ)$. So, on writing U for the number of solutions in question with $m_1 > m_2$, we find by symmetry that

$$\int_0^1 |f(\alpha)K(\alpha)|^2\, d\alpha \ll PZ + U. \tag{3.4}$$

For solutions counted by U, we put $z = m_1 + m_2$ and $w = m_1 - m_2$. We then have $1 < z \le 2P$ and $1 \le w \le P$, and the equation turns into

$$w(3z^2 + w^2) = 4\big(\phi(n_1) - \phi(n_2)\big).$$

Since the polynomial on the right hand side is divisible by $n_1 - n_2$, and the degree of our polynomial ϕ is at least two, we see that for each given pair of integers z and w, there are at most $O(\tau(w(3z^2 + w^2)))$ choices for n_1 and n_2 satisfying the latter equation, where $\tau(n)$ denotes the divisor function. Hence we have

$$U \ll \sum_{w \le P} \sum_{z \le 2P} \tau(w(3z^2 + w^2)) \le \sum_{w \le P} \tau(w) \sum_{z \le 2P} \tau(3z^2 + w^2). \tag{3.5}$$

In order to estimate the last double sum, we begin by evaluating the sum

$$U_1(w, X) = \sum_{\substack{z \le X \\ (z,w)=1}} \tau(3z^2 + w^2),$$

for a natural number $w \le X$ and a parameter $X \ge 2$. Plainly we have

$$U_1(w, X) \le 2 \sum_{\substack{z \le X \\ (z,w)=1}} \sum_{\substack{d \mid (3z^2+w^2) \\ d \le \sqrt{3z^2+w^2}}} 1 \le 2 \sum_{d \le 2X} \sum_{\substack{z \le X \\ (z,w)=1 \\ 3z^2+w^2 \equiv 0 \pmod{d}}} 1.$$

In the last innermost sum, we sort z according to its residue class modulo d, noting that in view of the summation conditions, every z necessarily belongs to a reduced residue class modulo d. Thus we have

$$U_1(w, X) \ll \sum_{d \le 2X} \sum_{\substack{1 \le a \le d \\ (a,d)=1 \\ 3a^2 \equiv -w^2 \pmod{d}}} \sum_{\substack{z \le X \\ z \equiv a \pmod{d}}} 1 \ll X \sum_{d \le 2X} \frac{a_w(d)}{d}, \tag{3.6}$$

where, as is apparent, we denote by $a_w(d)$ the number of integers a satisfying $1 \le a \le d$, with $(a, d) = 1$ and $3a^2 \equiv -w^2 \pmod{d}$.

One may swiftly confirm that $a_w(d)$ is multiplicative with respect to d. So we naturally consider the case where d is a prime power, and hereafter reserve the letter p to denote a prime. Then we may easily recognise that $a_w(p^\nu) = 0$ in the following three cases: (i) $p = 2$ and

$\nu \geq 3$, (ii) $p = 3$ and $\nu \geq 2$, (iii) $p|w$, $p \neq 3$ and $\nu \geq 1$. Moreover, by the standard theory of quadratic residues, when $p \nmid w$, $p > 3$ and $\nu \geq 1$, we may express $a_w(p^\nu)$ using the Legendre symbol as

$$a_w(p^\nu) = 1 + \left(\frac{-3}{p}\right) = 1 + \left(\frac{p}{3}\right).$$

Hence, from (3.6), we derive the estimate

$$U_1(w, X) \ll X \prod_{p \leq 2X} \left(1 + \sum_{\nu=1}^{\infty} \frac{a_w(p^\nu)}{p^\nu}\right) \ll X \prod_{\substack{p \leq 2X \\ p \equiv 1 \pmod 3}} \left(1 + \frac{2}{p}\right).$$

Since we know that

$$\sum_{\substack{p \leq 2X \\ p \equiv 1 \pmod 3}} \frac{1}{p} = \frac{1}{2} \log\log X + O(1),$$

we have

$$U_1(w, X) \ll X \exp\left(2 \sum_{\substack{p \leq 2X \\ p \equiv 1 \pmod 3}} \frac{1}{p}\right) \ll X \log X.$$

Having acquired the last bound for $U_1(w, X)$, we go back to (3.5), and sort the double sum according to the value of $l = (w, z)$, putting $w = lw'$ and $z = lz'$. Then, recalling in addition a basic estimate for the mean value of the divisor function, we obtain

$$\begin{aligned} U &\ll \sum_{l \leq P} \tau(l)^3 \sum_{w' \leq P/l} \tau(w') U_1\left(w', \frac{2P}{l}\right) \\ &\ll P(\log P) \sum_{l \leq P} \frac{\tau(l)^3}{l} \sum_{w' \leq P/l} \tau(w') \\ &\ll P^2 (\log P)^2 \sum_{l \leq P} \frac{\tau(l)^3}{l^2} \ll P^2 (\log P)^2. \end{aligned}$$

The estimate (3.1) now follows at once from (3.4).

To establish (3.2), we estimate U in a different way. Let $\mathcal{B}(P)$ be the set of natural numbers b with the property that there exist at least two distinct integers m_1 and m_2 satisfying $b = m_1^3 - m_2^3$, with $1 \leq m_2 < m_1 \leq P$. It is a consequence of work of Heath-Brown [9] that there are at most $O(P^{4/3+\varepsilon})$ solutions of the diophantine equation $m_1^3 - m_2^3 = m_3^3 - m_4^3$, subject to $1 \leq m_j \leq P$ $(1 \leq j \leq 4)$, and for which $m_1 \neq m_2$ and $(m_1, m_2) \neq (m_3, m_4)$. From this we deduce that the cardinality of $\mathcal{B}(P)$ is $O(P^{4/3+\varepsilon})$.

Now, recalling that the polynomial ϕ has degree at least two, and that $\phi(n_1)-\phi(n_2)$ is divisible by n_1-n_2, one finds via an elementary estimate for the divisor function that for each b with $1\le b\le P^3$, there are $O(P^\varepsilon)$ integers n_1 and n_2 satisfying $b=\phi(n_1)-\phi(n_2)$. A similar assertion may be confirmed concerning the equation $b=m_1^3-m_2^3$. Consequently, the number of solutions of (3.3) with $\phi(n_1)-\phi(n_2)\in\mathcal{B}(P)$ is $O(P^{4/3+\varepsilon})$. On the other hand, by the definition of $\mathcal{B}(P)$, for any n_1, $n_2\in\mathcal{Z}$ with $0<\phi(n_1)-\phi(n_2)\notin\mathcal{B}(P)$, there is at most one pair of integers m_1 and m_2 satisfying (3.3). Thus the number of solutions counted by U with $\phi(n_1)-\phi(n_2)\notin\mathcal{B}(P)$ cannot exceed Z^2. We may therefore conclude that $U\ll P^{4/3+\varepsilon}+Z^2$, and in view of (3.4), we obtain the estimate (3.2). This completes the proof of the lemma.

The proof of Theorem 1. We use the notation introduced in the preamble to Lemma 1, and set $k=3$ and $d_\phi=2$, so that (2.1) yields $P^{3/2}\ll x\ll P^{3/2}$. We aim to show that for any fixed small positive number δ, one has

$$\sum_{n\le x}|R_{6,3}(\phi(n);\mathfrak{m})|^2\ll P^{15/2}(\log P)^{\delta-3/2}, \tag{3.7}$$

and

$$\sum_{n\le x}|R_{7,3}(\phi(n);\mathfrak{m})|^4\ll P^{35/2}(\log P)^{\delta-15/2}. \tag{3.8}$$

On making use of the conclusion of Lemma 2, parts (i) and (ii) of Theorem 1 follow from the respective bounds (3.7) and (3.8).

For $s=6$ or 7, and for $n\le x$, we define the complex numbers η_n by means of the equation

$$\begin{aligned}|R_{s,3}(\phi(n);\mathfrak{m})|&=\left|\int_{\mathfrak{m}}f(\alpha)^s e(-\alpha\phi(n))\,d\alpha\right|\\&=\eta_n\int_{\mathfrak{m}}f(\alpha)^s e(-\alpha\phi(n))\,d\alpha,\end{aligned}$$

unless $R_{s,3}(\phi(n);\mathfrak{m})=0$, in which case we take $\eta_n=0$. Plainly, one always has $|\eta_n|\le 1$. Further, for $T>0$, we define

$$\mathcal{Z}_s(T)=\{n\le x\ :\ T<|R_{s,3}(\phi(n);\mathfrak{m})|\le 2T\},$$

write $Z_s(T)$ for the cardinality of $\mathcal{Z}_s(T)$, and introduce the function

$$K_{T,s}(\alpha)=\sum_{n\in\mathcal{Z}_s(T)}\eta_n e(\alpha\phi(n)).$$

On recalling (2.2), we then have

$$Z_s(T)T \ll \sum_{n\in\mathcal{Z}_s(T)} |R_{s,3}(\phi(n);\mathfrak{m})| = \int_{\mathfrak{m}} f(\alpha)^s K_{T,s}(-\alpha)\,d\alpha,$$

from which, by applying Schwarz's inequality, we infer that

$$(3.9)\quad Z_s(T)T \ll \big(\sup_{\alpha\in\mathfrak{m}}|f(\alpha)|\big)^{s-5}\left(\int_{\mathfrak{m}}|f(\alpha)|^8\,d\alpha\right)^{1/2} \times\left(\int_0^1 |f(\alpha)K_{T,s}(\alpha)|^2\,d\alpha\right)^{1/2}.$$

We next note that by incorporating the bounds for Hooley's Δ-function supplied by Hall and Tenenbaum [8] into the proof of Lemma 1 of Vaughan [11], one may confirm the estimate

$$\sup_{\alpha\in\mathfrak{m}}|f(\alpha)| \ll P^{3/4}(\log P)^{1/4+\varepsilon}.$$

In addition, we recall that Boklan [1] showed that

$$(3.10)\qquad \int_{\mathfrak{m}}|f(\alpha)|^8 d\alpha \ll P^5(\log P)^{\varepsilon-3}.$$

Substituting these results into (3.9) together with the estimate (3.1) of Lemma 3, we find that

$$Z_s(T)T \ll P^{3(s-1)/4}Z_s(T)^{1/2}(\log P)^{(s-11)/4+\varepsilon} + P^{(3s-1)/4}(\log P)^{(s-7)/4+\varepsilon},$$

whence

$$(3.11)\quad Z_s(T) \ll P^{3(s-1)/2}(\log P)^{(s-11)/2+\varepsilon}T^{-2} + P^{(3s-1)/4}(\log P)^{(s-7)/4+\varepsilon}T^{-1}.$$

Using the estimate (3.2) of Lemma 3 in place of (3.1) here, we also deduce from (3.9) that

$$Z_s(T)T \ll P^{3(s-1)/4}Z_s(T)^{1/2}(\log P)^{(s-11)/4+\varepsilon} + P^{3s/4-7/12+\varepsilon} + P^{(3s-5)/4}(\log P)^{(s-11)/4+\varepsilon}Z_s(T).$$

Let δ be any fixed positive number, and write

$$T_s = P^{(3s-5)/4}(\log P)^{(s-11+\delta)/4}.$$

Then the final term on the right hand side of the last inequality is irrelevant in circumstances in which $T \geq T_s$, because we may suppose that ε is sufficiently small in comparison with δ. Therefore, provided that $T \geq T_s$, we have

$$Z_s(T) \ll P^{3(s-1)/2}(\log P)^{(s-11)/2+\varepsilon}T^{-2} + P^{3s/4-7/12+\varepsilon}T^{-1}. \tag{3.12}$$

Now we concentrate on the proof of part (i) of Theorem 1. First we appeal to (3.12) with $s = 6$, and obtain

$$\begin{aligned}\sum_{n\in\mathcal{Z}_6(T)} |R_{6,3}(\phi(n);\mathfrak{m})|^2 &\ll Z_6(T)T^2 \\ &\ll P^{15/2}(\log P)^{\varepsilon-5/2} + P^{47/12+\varepsilon}T,\end{aligned} \tag{3.13}$$

provided that $T \geq T_6 = P^{13/4}(\log P)^{(\delta-5)/4}$.

Next we derive an upper bound for $|R_{6,3}(\phi(n);\mathfrak{m})|$. Recalling (3.10) and the bound

$$\int_0^1 |f(\alpha)|^4\, d\alpha \ll P^2 \tag{3.14}$$

that follows from Hooley [10], an application of Schwarz's inequality reveals that

$$\begin{aligned}|R_{6,3}(\phi(n);\mathfrak{m})| &\leq \left(\int_0^1 |f(\alpha)|^4\, d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}} |f(\alpha)|^8\, d\alpha\right)^{1/2} \\ &\ll P^{7/2}(\log P)^{\varepsilon-3/2}.\end{aligned}$$

So $\mathcal{Z}_6(T)$ is empty for $T \geq P^{7/2}$. Therefore, putting $T = 2^l T_6$ and summing the inequality (3.13) over integers l for which $T_6 \leq T \leq P^{7/2}$, we obtain

$$\sum_{\substack{n\leq x\\ |R_{6,3}(\phi(n);\mathfrak{m})|\geq T_6}} |R_{6,3}(\phi(n);\mathfrak{m})|^2 \ll P^{15/2}(\log P)^{\varepsilon-3/2}. \tag{3.15}$$

To treat the case where $T \leq T_6$, we use (3.11), and find that

$$\begin{aligned}\sum_{n\in\mathcal{Z}_6(T)} |R_{6,3}(\phi(n);\mathfrak{m})|^2 &\ll Z_6(T)T^2 \\ &\ll P^{15/2}(\log P)^{\varepsilon-5/2} + P^{17/4}(\log P)^{\varepsilon-1/4}T.\end{aligned}$$

Putting $T = 2^l T_6$ again, and summing up the latter inequality this time for all integers l with $1 \leq T < T_6$, we secure the bound

$$\sum_{\substack{n\leq x\\ 2<|R_{6,3}(\phi(n);\mathfrak{m})|\leq T_6}} |R_{6,3}(\phi(n);\mathfrak{m})|^2 \ll P^{15/2}(\log P)^{\varepsilon+\delta/4-3/2}. \tag{3.16}$$

As the terms with $|R_{6,3}(\phi(n);\mathfrak{m})| \le 2$ are negligible, the desired bound (3.7) is a consequence of (3.15) and (3.16), and part (i) of Theorem 1 follows from Lemma 2.

We next turn to part (ii) of Theorem 1. Applying (3.12) with $s=7$, we find that

$$\begin{aligned}\sum_{n\in\mathcal{Z}_7(T)} |R_{7,3}(\phi(n);\mathfrak{m})|^4 &\ll Z_7(T)T^4 \\ &\ll P^9(\log P)^{\varepsilon-2}T^2 + P^{14/3+\varepsilon}T^3, \end{aligned} \tag{3.17}$$

provided that $T \ge T_7 = P^4(\log P)^{\delta/4-1}$.

We can obtain an upper bound for $|R_{7,3}(\phi(n);\mathfrak{m})|$ from (3.10) and (3.14) as before, but here we expend a little effort to establish a slightly sharper bound. Following Vaughan [11], we define $\mathcal{J}$ to be the set of ordered pairs (m_1,m_2) such that $m_1\le P$, $m_2\le P$, the greatest common divisor of m_1 and m_2 is at most $(\log P)^{80}$, and neither m_1 nor m_2 has a prime divisor p with $(\log P)^{80} < p \le P^{1/7}$. Then, by the methods of Vaughan [11] (see the argument on pp. 137-138 of [11]), one obtains

$$\int_{\mathfrak{m}} \Bigg| \sum_{\substack{m_1,m_2\le P\\ (m_1,m_2)\notin\mathcal{J}}}\sum e(\alpha m_1^3-\alpha m_2^3)\Bigg| |f(\alpha)|^6\,d\alpha \ll P^5(\log P)^{-18}.$$

On combining this bound with (3.10) and (3.14) within an application of Hölder's inequality, one finds that

$$\begin{aligned}\int_{\mathfrak{m}} \Bigg| \sum_{\substack{m_1,m_2\le P\\ (m_1,m_2)\notin\mathcal{J}}}\sum e(\alpha m_1^3-\alpha m_2^3)\Bigg|^{1/2} |f(\alpha)|^6\,d\alpha \\ \ll \left(\int_0^1 |f(\alpha)|^4\,d\alpha\right)^{1/4}\left(\int_{\mathfrak{m}} |f(\alpha)|^8\,d\alpha\right)^{1/4} \\ \times \left(\int_{\mathfrak{m}} \Bigg| \sum_{\substack{m_1,m_2\le P\\ (m_1,m_2)\notin\mathcal{J}}}\sum e(\alpha m_1^3-\alpha m_2^3)\Bigg| |f(\alpha)|^6\,d\alpha\right)^{1/2} \\ \ll P^{17/4}(\log P)^{-9}. \end{aligned} \tag{3.18}$$

Next, by appealing to the linear sieve as in the conclusion of the proof of Theorem B of Vaughan [11], one may see that there are at most $O(P(\log P)^{\varepsilon-1})$ natural numbers up to P without prime divisors in the interval $((\log P)^{80}, P^{1/7}]$. By considering the underlying diophantine equation, we deduce from Lemma 2 of Vaughan [11] that

$$\int_0^1 \Big| \sum\sum_{(m_1,m_2)\in\mathcal{J}} e(\alpha m_1^3 - \alpha m_2^3)\Big|^2 \, d\alpha \ll P^2 (\log P)^{\varepsilon - 2}.$$

A swift application of Hölder's inequality therefore leads from (3.10) to the estimate

$$\begin{aligned} &\int_{\mathfrak{m}} \Big| \sum\sum_{(m_1,m_2)\in\mathcal{J}} e(\alpha m_1^3 - \alpha m_2^3)\Big|^{1/2} |f(\alpha)|^6 \, d\alpha \\ &\ll \Big(\int_0^1 \Big| \sum\sum_{(m_1,m_2)\in\mathcal{J}} e(\alpha m_1^3 - \alpha m_2^3)\Big|^2 \, d\alpha \Big)^{1/4} \Big(\int_{\mathfrak{m}} |f(\alpha)|^8 \, d\alpha \Big)^{3/4} \\ &\ll P^{17/4} (\log P)^{\varepsilon - 11/4}. \qquad (3.19) \end{aligned}$$

Since

$$|f(\alpha)|^2 = \sum\sum_{\substack{m_1,m_2\le P \\ (m_1,m_2)\notin\mathcal{J}}} e(\alpha m_1^3 - \alpha m_2^3) + \sum\sum_{(m_1,m_2)\in\mathcal{J}} e(\alpha m_1^3 - \alpha m_2^3),$$

it follows from (3.18) and (3.19) that

$$|R_{7,3}(\phi(n);\mathfrak{m})| \le \int_{\mathfrak{m}} |f(\alpha)|^7 \, d\alpha \ll P^{17/4} (\log P)^{\varepsilon - 11/4}, \qquad (3.20)$$

whence $\mathcal{Z}_7(T)$ is empty for $T \ge P^{17/4}(\log P)^{(\delta-11)/4}$. Therefore, putting $T = 2^l T_7$ and summing (3.17) over integers l for which $T_7 \le T \le P^{17/4}(\log P)^{(\delta-11)/4}$, we obtain

$$\sum_{\substack{n\le x \\ |R_{7,3}(\phi(n);\mathfrak{m})|\ge T_7}} |R_{7,3}(\phi(n);\mathfrak{m})|^4 \ll P^{35/2} (\log P)^{\varepsilon + (\delta - 15)/2}. \qquad (3.21)$$

For $T \le T_7$, we use (3.11) with $s = 7$, and find that

$$\sum_{n\in\mathcal{Z}_7(T)} |R_{7,3}(\phi(n);\mathfrak{m})|^4 \ll Z_7(T) T^4 \ll P^9 T_7^2 + P^{5+\varepsilon} T_7^3 \ll P^{17+\varepsilon}.$$

It follows easily from this that the contribution of numbers n with $|R_{7,3}(\phi(n);\mathfrak{m})| \le T_7$ to the left hand side of (3.8) is at most $O(P^{17+\varepsilon})$. The upper bound (3.8) now follows from (3.21), and then part (ii) of Theorem 1 follows by applying Lemma 2. In this way, we complete the proof of Theorem 1.

The proof of Theorem 2. We adopt the notation employed in the proof of Theorem 1 above, fixing $s = 7$, save that ϕ is now supposed to be an integral cubic polynomial. Since ϕ is cubic in the situation at hand, it follows from (2.1) that

$$P \ll x \ll P. \tag{3.22}$$

This change has no effect, in particular, on the argument leading to (3.12), and we may therefore infer that for $T \geq T_7 = P^4(\log P)^{\delta/4-1}$, one has

$$\sum_{n \in \mathcal{Z}_7(T)} |R_{7,3}(\phi(n);\mathfrak{m})|^2 \ll Z_7(T)T^2$$
$$\ll P^9(\log P)^{\varepsilon-2} + P^{14/3+\varepsilon}T.$$

On noting that (3.20) remains valid in the current situation, we may put $T = 2^l T_7$ and sum the last inequality over integers l for which $T_7 \leq T < P^{17/4}$. Thus we deduce that

$$\sum_{\substack{n \leq x \\ |R_{7,3}(\phi(n);\mathfrak{m})| \geq T_7}} |R_{7,3}(\phi(n);\mathfrak{m})|^2 \ll P^9(\log P)^{\varepsilon-1}.$$

Moreover, in view of (3.22), it is apparent that

$$\sum_{\substack{n \leq x \\ |R_{7,3}(\phi(n);\mathfrak{m})| \leq T_7}} |R_{7,3}(\phi(n);\mathfrak{m})|^2 \ll xT_7^2 \ll P^9(\log P)^{\delta/2-2}.$$

Consequently, we find that

$$\sum_{n \leq x} |R_{7,3}(\phi(n);\mathfrak{m})|^2 \ll P^9(\log P)^{\varepsilon-1}.$$

On noting that in the current situation, we have $d_\phi = 3$, and recalling (3.22), the conclusion of Theorem 2 follows from Lemma 2.

4. Waring's problem for larger exponents.

We close this paper by proving Theorem 3 using an argument similar to that employed in the preceding section. We again adopt the notation introduced in the preamble to Lemma 1, but here we suppose that k is a sufficiently large integer, and set $d_\phi = 2$. Thus ϕ is an integral quadratic polynomial with positive leading coefficient, so that from (2.1) one has $P^{k/2} \ll x \ll P^{k/2}$. By virtue of Lemma 2, the asymptotic formula (1.6) will follow once we have established an estimate of the shape

$$\sum_{n \leq x} |R_{s,k}(\phi(n);\mathfrak{m})|^h \ll P^{h(s-k)+k/2-\xi}, \tag{4.1}$$

for some positive number ξ.

As before, we define the complex numbers η_n by means of the equation

$$
\begin{aligned}
|R_{s,k}(\phi(n);\mathfrak{m})| &= \left| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha\phi(n))\, d\alpha \right| \\
&= \eta_n \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha\phi(n))\, d\alpha,
\end{aligned}
$$

unless $R_{s,k}(\phi(n);\mathfrak{m}) = 0$, in which case we take $\eta_n = 0$. Thus, for each natural number n, we have $|\eta_n| \le 1$. Also, for $T > 0$, we introduce the set

$$\mathcal{Z}(T) = \{ n \le x \,:\, T < |R_{s,k}(\phi(n);\mathfrak{m})| \le 2T \},$$

we write $Z(T)$ for the cardinality of $\mathcal{Z}(T)$, and we define the function

$$K_T(\alpha) = \sum_{n \in \mathcal{Z}(T)} \eta_n e(\alpha\phi(n)).$$

With these definitions in hand, an application of Hölder's inequality reveals that

$$
\begin{aligned}
Z(T)T \le \sum_{n \in \mathcal{Z}(T)} |R_{s,k}(\phi(n);\mathfrak{m})| &= \int_{\mathfrak{m}} f(\alpha)^s K_T(-\alpha) d\alpha \\
&\ll \left(\int_0^1 |K_T(\alpha)|^{2h} d\alpha \right)^{1/(2h)} \left(\int_{\mathfrak{m}} |f(\alpha)|^{2hs/(2h-1)} d\alpha \right)^{1-1/(2h)}. \qquad (4.2)
\end{aligned}
$$

In the first part of this series of papers [3], we established an estimate tantamount to

$$\int_0^1 |K_T(\alpha)|^4 d\alpha \ll Z(T)^2 P^{\varepsilon}$$

(see (3.16) of [3]). Making use of this inequality and the trivial bound $|K_T(\alpha)| \le Z(T)$, we derive for $h \ge 2$ the upper bound

$$\int_0^1 |K_T(\alpha)|^{2h} d\alpha \le Z(T)^{2h-4} \int_0^1 |K_T(\alpha)|^4 d\alpha \ll Z(T)^{2h-2} P^{\varepsilon}. \qquad (4.3)$$

Moreover, one may apply the conclusions of Ford [7] to show that

$$\int_{\mathfrak{m}} |f(\alpha)|^u \, d\alpha \ll P^{u-k-\xi}, \qquad (4.4)$$

for some $\xi > 0$, provided that $u > k^2(\log k + \log\log k + O(1))$. Indeed, by the computation leading to the estimate (4.25) of [4], we see that provided one has

$$u > 2k \left\lceil \frac{1}{2} k(\log k + \log\log k + 1) \right\rceil + 6k^2,$$

then (4.4) holds with $\xi = (5\log k)^{-1}$. Here, as usual, we have written $\lceil X\rceil$ for the least integer n with $n \geq X$. In the application at hand, we take $u = 2hs/(2h-1)$. Then for large values of k, the last condition is satisfied if, for example, one has

$$s \geq \Big(1 - \frac{1}{2h}\Big)k^2(\log k + \log\log k + 8). \tag{4.5}$$

We insert the bounds (4.3) and (4.4) into (4.2), and with a modicum of computation we infer that

$$Z(T)^{1/h}T \ll P^{s-(2h-1)(k+\xi)/(2h)+\varepsilon}.$$

From this we deduce that

$$\sum_{n\in\mathcal{Z}(T)} |R_{s,k}(\phi(n);\mathfrak{m})|^h \ll Z(T)T^h \ll P^{sh-kh+k/2-(h-1/2)\xi+\varepsilon}.$$

On noting the trivial bound $|R_{s,k}(\phi(n);\mathfrak{m})| \leq P^s$, and observing that the terms with $|R_{s,k}(\phi(n);\mathfrak{m})| \leq 1$ are negligible, the desired bound (4.1) follows for each $h \geq 2$, and exponents s satisfying (4.5), on putting $T = 2^l$, and summing this inequality over integers l with $0 \leq l \leq 2s\log P$. Hence, in view of Lemma 2, the conclusion of Theorem 3 follows at once.

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