ADDITIVE REPRESENTATION IN THIN SEQUENCES, VIII: DIOPHANTINE INEQUALITIES IN REVIEW

Jörg Brüdern

Institut für Algebra und Zahlentheorie, Universität Stuttgart, 70511 Stuttgart, Germany
E-mail: bruedern@mathematik.uni-stuttgart.de

Koichi Kawada

Department of Mathematics, Faculty of Education, Iwate University, Morioka, 020–8550 Japan
E-mail: kawada@iwate-u.ac.jp

Trevor D. Wooley

School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom
E-mail: matdw@bristol.ac.uk

Recent developments in the theory of diophantine inequalities and the Davenport-Heilbronn method are discussed and then directed toward specific inequalities of definite character. Special emphasis is on the value-distribution of diagonal forms near thin test sequences.

1. Theme and results

1.1. *Diophantine inequalities*

The focus of our attention in this survey article is the distribution of the values of the diagonal form

$$\lambda_1 x_1^k + \lambda_2 x_2^k + \cdots + \lambda_s x_s^k, \tag{1.1}$$

as $x_1, \dots, x_s$ range over $\mathbb{Z}$ or an interesting subset thereof; here $s \geq 2$ and $k \geq 1$ are given integers and $\lambda_1, \dots, \lambda_s$ are non-zero real numbers. Our goal is twofold. In one direction we wish to emphasise recent developments in the analytic theory of diophantine inequalities, in another discuss the potential of methods developed in earlier papers of this series pertaining to this circle of ideas. Some basic principles are also readdressed herein, so that the paper should introduce the uninitiated reader to the subject.

When the polynomial (1.1) is a real multiple of a form with integer coefficients, its values are discrete and can be studied by classical methods as well as the techniques developed in this series. The complementary case, in which (1.1) is not a multiple of a rational form, arises when at least one of the ratios λ_1/λ_j $(2 \le j \le s)$ is irrational, and by renumbering the variables, we shall from now on suppose that λ_1/λ_2 is irrational. In this situation, and when $s \ge k+1$, one expects that the values of (1.1) are dense on the real line unless k is even and all λ_j have the same sign. In the latter case, one might hope that the gaps between these values shrink to zero as they approach infinity. When $k = 1$, this is classical territory as long as the variables x_j are allowed to vary over the integers, but, for example, much less is known about the distribution of $\lambda_1 p_1 + \lambda_2 p_2$ when p_1, p_2 denote primes. When $k = 2$, an affirmative theorem is available for indefinite quadratic forms from the work of Margulis [28] (see also [19]), and also for the definite case when $s \ge 5$. This was shown by Götze [25], following the pivotal contribution by Bentkus and Götze [2]. Their work may be viewed as a formidable refinement of a Fourier transform method developed by Davenport and Heilbronn [17]. This very general method is a variant for diophantine inequalities of the famous circle method of Hardy and Littlewood that delivered enormous insight into the labyrinth of diophantine equations during the last century. Davenport and Heilbronn themselves studied the indefinite case $k = 2$ of (1.1) and showed that the values taken at integer points are dense on the real line. Their pioneering paper was the igniting spark for much work on related questions. It would take us too far afield to sketch the development of the subject at large, so instead we follow a fruitful tradition in additive number theory, where new ideas have often been tested on Waring's problem for cubes or on Goldbach's problem. Thus, we now concentrate on the case $k = 3$ in (1.1), and later move on to the value distribution of $\lambda_1 p_1 + \lambda_2 p_2$, although the methods that we develop apply in a much wider context.

1.2. *Additive cubic forms*

We discuss a slightly narrower problem from now on, by enforcing a kind of definiteness. Let $\lambda_1, \dots, \lambda_s$ denote *positive* real numbers, to be considered as fixed once and for all. For $0 < \tau \le 1$ and $\nu > 0$, let $\rho_s(\tau, \nu)$ be the number of solutions of the inequality

$$|\lambda_1 x_1^3 + \lambda_2 x_2^3 + \cdots + \lambda_s x_s^3 - \nu| < \tau \tag{1.2}$$

in natural numbers x_j. Note that for any solution counted here one has $x_j \ll \nu^{1/3}$. In accordance with our earlier remarks, we still expect the gaps among large values of $\lambda_1 x_1^3 + \lambda_2 x_2^3 + \cdots + \lambda_s x_s^3$ to shrink to zero provided that $s \geq 4$ and λ_1/λ_2 is irrational. One anticipates that $\rho_s(\tau, \nu)$ should be large when τ is fixed and ν is large. The work of Freeman [21], [22] applies to this problem, and so does the refinement by Wooley [44], and it is implicit in the latter that for τ fixed, one has

$$\rho_7(\tau, \nu) \gg \nu^{4/3}, \quad \rho_8(\tau, \nu) = c\nu^{5/3}(1 + o(1)). \tag{1.3}$$

Here c is a certain positive constant depending only on τ and the coefficients λ_j. Freeman and Wooley consider in detail a different scenario: ν is fixed, one counts solutions of (1.2) in integers x_j with $|x_j| \leq X$, and examines the growth for $X \to \infty$. Little change is necessary to derive the "definite" versions (1.3) by the arguments of Wooley [44], but older routines based on the original work of Davenport and Heilbronn [17] do not apply. Instead one would find only that (1.2) has infinitely many solutions in *integers* x_j.

Thus far, the situation is in direct correspondence to what is known when the cubic form in (1.2) is a multiple of a rational form. Focusing on this case temporarily, when τ is sufficiently small, the inequality (1.2) reduces to an equation. Hence there is no loss in assuming here that all λ_j are natural numbers, and that the equation is

$$\lambda_1 x_1^3 + \lambda_2 x_2^3 + \cdots + \lambda_s x_s^3 = \nu. \tag{1.4}$$

Let $\rho_s(\nu)$ denote the number of its solutions with $x_j \in \mathbb{N}$. The methods of Vaughan [33], [34] provide the lower bound $\rho_7(\nu) \gg \mathfrak{S}_7(\nu)\nu^{4/3}$, and the asymptotic formula $\rho_8(\nu) = C\mathfrak{S}_8(\nu)\nu^{5/3}(1 + o(1))$, where $\mathfrak{S}_s(\nu)$ is the singular series associated with (1.4), and C is a positive constant depending only on the λ_j. Similar formulae are expected when $s \geq 4$, and are at least implicitly known *on average*. In fact, the envisaged formula for $\rho_4(\nu)$ holds for all but $O(N(\log N)^{\varepsilon - 3})$ of the natural numbers ν not exceeding N. This much follows from the work of Vaughan [33] and Boklan [3]. For an analogue in the irrational case, one must first address the question of how one should average over the now real number ν. One could choose a discrete sequence of test points that are suitably spaced, and then count how often an asymptotic formula for $\rho_4(\tau, \nu)$ fails. Alternatively, one may estimate the measure of all such real $\nu \in [1, N]$. Only very recently Parsell and Wooley [31] proved that this measure is $o(N)$. As an illustration of the averaging process, we improve their estimate when λ_1/λ_2 is an algebraic irrational. The result, which we deduce as a consequence of Theorem 2.2 in §2.3 below, fully reflects the current state of knowledge for forms with integer coefficients, but it appears difficult to do equally well under the sole assumption that λ_1/λ_2 is irrational.

Theorem 1.1. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote positive real numbers with λ_1/λ_2 irrational and algebraic. Then, whenever $(\log N)^{-3} < \tau \le 1$, one has*

$$\int_0^N \left| \rho_4(\tau,\nu) - \frac{2\Gamma(\frac{4}{3})^3 \tau \nu^{1/3}}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/3}} \right|^2 \mathrm{d}\nu \ll \tau^{3/2} N^{5/3} (\log N)^{\varepsilon - 3/2}.$$

Previous articles within this series exposed methods for testing the conjectural behaviour of counting functions such as $\rho_s(\nu)$ when ν varies over a thin sequence, for example the values of a polynomial. A typical result is contained in Theorem 1.1 of III* that we now recall. It will be convenient to describe a polynomial $\phi \in \mathbb{R}[t]$ of degree $d \ge 1$ as a *positive polynomial* if its leading coefficient is positive. If $\phi \in \mathbb{Q}[t]$ and $\phi(n)$ is integral for all $n \in \mathbb{Z}$, then ϕ is an *integral polynomial*. Let $r_s(n)$ be the number of positive solutions of $n = x_1^3 + x_2^3 + \ldots + x_s^3$. Then, for any $0 < \delta \le \frac{1}{2}$, the inequality

$$|r_6(\nu) - \Gamma(\tfrac{4}{3})^6 \mathfrak{S}_6(\nu)\nu| > \nu(\log \nu)^{-\delta}$$

can hold for no more than $O(N(\log N)^{2\delta - 5/2 + \varepsilon})$ of the values $\nu = \phi(n)$ with $n \le N$ assumed by a positive integral quadratic polynomial ϕ. There is no difficulty in extending this to forms with positive integral coefficients.

Our primary concern in the later chapters of this paper is to describe methods that allow one to derive similar results in the context of diophantine inequalities. Later we will comment on some of the difficulties that arise, and we shall find the desired generalisation not as straightforward as one might hope. A conclusion for $\rho_6(\tau,\nu)$ of strength comparable to the aforementioned theorem on $r_6(\nu)$ is contained in the next result, the proof of which may be found in §5.2.

Theorem 1.2. *Let ϕ denote a positive integral quadratic polynomial, and let $\lambda_1, \ldots, \lambda_6$ denote positive real numbers with λ_1/λ_2 irrational. Also, let $0 < \tau \le 1$. Then there exists a function $\xi(\nu)$, with $\xi(\nu) = o(1)$ as $\nu \to \infty$, such that the inequality*

$$|\rho_6(\tau,\nu) - 2\Gamma(\tfrac{4}{3})^6 (\lambda_1 \cdots \lambda_6)^{-1/3} \tau\nu| > \nu\xi(\nu)$$

holds for at most $O(N(\log N)^{\varepsilon - 5/2})$ of the positive values $\nu = \phi(n)$ with $1 \le n \le N$.

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*Here and later we refer to our papers "Additive representation in thin sequences" by their numeral within the series, I–VII. Hence, III refers to [9], for example.

One might object that although it is rather natural to average over the values of an integral polynomial in the case of diophantine equations, this is not adequate for inequalities, and one should take the values of a real polynomial as test points, or even a monotone sequence with a certain rate of growth. In principle, our methods still apply in this wider context, but some techniques such as certain divisor estimates do no longer have their full impact on the problem at hand, and the ensuing results are sometimes considerably weaker. We illustrate this in §§6.2 and 6.3 with an analysis of following example.

Theorem 1.3. *Let ϕ denote a positive quadratic polynomial, and let $\lambda_1, \ldots, \lambda_6$ denote positive real numbers for which λ_1/λ_2 is irrational. Fix $0 < \tau \le 1$. Then, there exists a real number $c > 0$ such that, for all but $O(N^{23/27})$ of the integers $n \in [1, N]$, one has $\rho_6(\tau, \phi(n)) \ge c\phi(n)$.*

We have not been able to establish the expected asymptotic formula almost always when ϕ is not an integral polynomial. Theorem 1.3 should also be compared with Theorem 1.1 of I where the exceptional set is shown to be $O(N^{19/28})$ when $\rho_6(\tau, \nu)$ is replaced by $r_6(\nu)$.

Similar results for forms in five variables are not yet available. This applies even to the simplest examples in the rational case: it is not known whether almost all squares are the sum of five positive cubes. In such situations our methods can sometimes be turned toward a lower bound estimate. The idea is discussed in detail in IV, and then used to show that when ϕ is a positive integral quadratic polynomial, then amongst the integers n with $1 \le n \le N$, the equation $\phi(n) = x_1^3 + \ldots + x_5^3$ has solutions with $x_j \in \mathbb{N}$ for at least $N^{129/136}$ values of n (see Theorem 1.1 of IV). In §6.4 we prove a result of similar flavour.

Theorem 1.4. *Let ϕ denote a positive quadratic polynomial, and let $\lambda_1, \ldots, \lambda_5$ be positive real numbers with λ_1/λ_2 irrational. Fix $0 < \tau \le 1$. Then $\rho_5(\tau, \phi(n)) \ge 1$ for at least $N^{3/4}$ natural numbers $n \in [1, N]$.*

Cubic forms in seven variables, in the rational case, have also been discussed in III and VII, although only in the context of sums of cubes. Moments of $r_s(n)$ over polynomial sequences are one of the objectives in VII, and in particular, Theorem 1 of VII contains an asymptotic formula for the sum

$$\sum_{n \le N} r_7(\phi(n))^2$$

when ϕ is a positive integral quadratic polynomial. The result coincides with the formula that arises from summing the leading term in the anticipated asymptotic expansion of $r_7(\nu)$. Similarly, under the assumptions of Theorem 1.3 (suitably adapted to the current context with seven variables), one can derive an asymptotic formula for

$$\sum_{n\le N} \rho_7(\tau,\phi(n))^2.$$

It suffices to follow the pattern laid out in VII, but the argument is simpler in the absence of a singular series, and we spare the reader any detail.

1.3. *Linear forms in primes*

We now turn to the value distribution of the binary form $\lambda_1 p_1 + \lambda_2 p_2$ with positive coefficients λ_1, λ_2 and large prime variables p_1, p_2. When λ_1/λ_2 is rational, we may as well suppose that $\lambda_1, \lambda_2 \in \mathbb{N}$, and one then wishes to solve $\lambda_1 p_1 + \lambda_2 p_2 = n$ for a given natural number n. A necessary condition is that $\lambda_1 x_1 + \lambda_2 x_2 \equiv n \bmod 2\lambda_1\lambda_2$ has a solution in integers x_1, x_2 both coprime to $2\lambda_1\lambda_2$ (the *congruence condition*). It is at least implicit in the work of Montgomery and Vaughan [29] and of Liu and Tsang [27] that the number of natural numbers $n \le N$ which satisfy the congruence condition, but have no representation in the form $n = \lambda_1 p_1 + \lambda_2 p_2$, does not exceed $O(N^{1-\delta})$, for some $\delta > 0$. Pintz has recently announced that one may take any $\delta < \frac{1}{3}$ here, at least when $\lambda_1 = \lambda_2 = 1$. A result of comparable strength is available for the irrational case $\lambda_1/\lambda_2 \notin \mathbb{Q}$ when this ratio is algebraic. This was observed by Brüdern, Cook and Perelli [6]. We illustrate the underlying idea in the second half of §2.3 with a related result. For $0 < \tau \le 1$ and $\nu > 0$, let $\sigma(\tau,\nu)$ denote the number of prime solutions to

$$|\lambda_1 p_1 + \lambda_2 p_2 - \nu| < \tau, \tag{1.5}$$

with each solution p_1, p_2 counted with weight $(\log p_1)(\log p_2)$.

Theorem 1.5. *Let λ_1, λ_2 denote positive real numbers such that λ_1/λ_2 is an algebraic irrational. Then, for any $A \ge 1$, the set of real numbers ν with $1 \le \nu \le N$, for which*

$$\left|\sigma(\tau,\nu) - \frac{2\tau\nu}{\lambda_1\lambda_2}\right| > \frac{\tau\nu}{(\log N)^A}, \tag{1.6}$$

has measure $O(\tau^{-1}N^{2/3+\varepsilon})$, uniformly in $0 < \tau \le 1$.

For comparison, Parsell [30] works under the weaker hypothesis that λ_1/λ_2 is irrational, and obtains a result that is essentially equivalent to

$$\int_0^N \left|\sigma(\tau,\nu) - \frac{2\tau\nu}{\lambda_1\lambda_2}\right|^2 \mathrm{d}\nu = o(N^3). \tag{1.7}$$

Our method also gives a proof of (1.7), as well as an improvement when λ_1/λ_2 is algebraic, but not by a power of N. Limitations arise from our current knowledge concerning the zeros of the Riemann zeta function. Thus, when λ_1/λ_2 is algebraic, one may apply the methods described below to confirm that the integral on the left hand side of (1.7) is $O(N^3\exp(-c\sqrt{\log N}))$ for some $c > 0$, and with only moderate extra effort one obtains a saving that corresponds to the sharpest one currently known in the error term for the prime number theorem.

Next, we investigate averages over polynomial sequences. Theorem 1 of II asserts that there is a constant $\delta > 0$ such that, for any positive integral polynomial ϕ of degree d, the number of even values of $\phi(n)$ with $1 \le n \le N$ that are not the sum of two primes does not exceed $O(N^{1-\delta/d})$. In the irrational case we require λ_1/λ_2 to be algebraic, and the conclusion is decidedly weaker. In §5.6 we sketch a proof of the following result.

Theorem 1.6. *Let λ_1, λ_2 denote positive real numbers such that λ_1/λ_2 is an algebraic irrational. Fix $0 < \tau \le 1$ and $A \ge 1$. Let ϕ denote a positive polynomial of degree d, and let $E_\phi(N)$ denote the number of integers n with $1 \le n \le N$ for which the inequality* (1.6) *holds with $\nu = \phi(n)$. Then, there is an absolute constant $\delta > 0$ such that*

$$E_\phi(N) \ll N^{1-\delta/(d\log d)}.$$

In the absence of the hypothesis that λ_1/λ_2 be algebraic, it seems difficult to establish a quantitative bound for $E_\phi(N)$, but proving that $E_\phi(N) = o(N)$ is straightforward. In chapter 4 we average over even thinner sets. Brüdern and Perelli [15] have a corresponding result on Goldbach's problem.

Theorem 1.7. *Let λ_1, λ_2 denote positive real numbers such that λ_1/λ_2 is an algebraic irrational. Fix $0 < \tau \le 1$ and $A \ge 1$. Let $1 < \gamma < \frac{3}{2}$, and write $\phi(t) = \exp((\log t)^\gamma)$. Let $E_\phi(N)$ be the number of natural numbers n with $1 \le n \le N$ for which the inequality* (1.6) *holds with $\nu = \phi(n)$. Then, there is a $\kappa > 0$ such that*

$$E_\phi(N) \ll N\exp(-\kappa(\log N)^{3-2\gamma}).$$

1.4. *Further applications*

Various other examples of averages over thin sequences can be found in I–VI, and one may extend most of them to diophantine inequalities along the lines indicated above. We single out two results from V and VI that concern the form (1.1) when the degree k is large. Theorem 1.5 of V shows that whenever $s \geq \frac{1}{2}k\log k + O(k\log\log k)$ and ϕ is a quadratic, positive and integral polynomial, then for almost all $n \in \mathbb{N}$ the number $\phi(n)$ is the sum of s positive k-th powers. This may be somewhat surprising, because one cannot do substantially better in the seemingly simpler problem in which the quadratic polynomial $\phi(n)$ is replaced by a linear one; the lower bound on s required here is again of the type $s \geq \frac{1}{2}k\log k + O(k\log\log k)$. This much is implicit in the work of Wooley [40].

Now suppose that $\lambda_1 x_1^k + \ldots + \lambda_s x_s^k$ is an irrational form with positive coefficients, and let $0 < \tau \leq 1$ and $\nu > 0$. Let $\rho_{k,s}(\tau,\nu)$ denote the number of solutions of the inequality

$$|\lambda_1 x_1^k + \lambda_2 x_2^k + \ldots + \lambda_s x_s^k - \nu| < \tau$$

in positive integers x_j (whence $\rho_{3,s} = \rho_s$ in the notation of §1.2). One can now combine the methods used to prove Theorem 1.2 in this paper with the strategy explained in V to confirm that whenever ϕ is a positive *integral* quadratic polynomial and τ is fixed, then for almost all n, one has $\rho_{k,s}(\tau,\phi(n)) > 0$, provided only that $s \geq \frac{1}{2}k\log k + O(k\log\log k)$. This is a proper analogue of the aforementioned result on Waring's problem. For a general real polynomial, as we discover in §6.7, the problem is more difficult.

Theorem 1.8. *Let $\lambda_1, \ldots, \lambda_s$ be positive real numbers, and suppose that λ_1/λ_2 is irrational. Let $0 < \tau \leq 1$. Let ϕ be a positive quadratic polynomial. Then there is a number $s_0(k)$, with $s_0(k) = \frac{3}{4}k\log k + O(k\log\log k)$, such that whenever $s \geq s_0(k)$, then for almost all n one has $\rho_{k,s}(\tau,\phi(n)) > 0$.*

Similar conclusions can be obtained when ϕ is a polynomial of degree $d \geq 3$, by a method akin to that used to establish Theorem 1.6.

One may also ask whether the form (1.1) takes values near an arithmetic sequence, such as the primes. This theme was discussed in VI, and we derive in §6.8 an analogue of Theorem 1 from that paper.

Theorem 1.9. *Suppose that all Dirichlet L-functions satisfy the Riemann hypothesis. Let λ_j be as in Theorem 1.8, and suppose that $s \geq \frac{8}{3}k+2$. Then, the integer parts of $\lambda_1 x_1^k + \lambda_2 x_2^k + \ldots + \lambda_s x_s^k$ are prime infinitely often for natural numbers x_j.*

1.5. *A related diophantine inequality*

A polynomial $\phi \in \mathbb{R}[t]$ is described as *irrational* if it is not a real multiple of an integral polynomial. It is in this case that our results are usually rather weaker than their equation counterparts in other papers of this series; for integral polynomials we experienced little difficulty in extending our ideas for averaging over polynomial values to diophantine inequalities. In part, this is due to the available estimates for the number of solutions of such inequalities as

$$\Big|\sum_{j=1}^{t}(\phi(n_j)-\phi(m_j))\Big|<1, \tag{1.8}$$

in natural numbers $n_j, m_j \le N$. These differ substantially according to whether ϕ is integral or irrational. Auxiliary bounds of this type are also relevant for a class of diophantine inequalities recently discussed by Freeman [24]. He considers a set of s non-zero positive polynomials

$$\phi_j(t)=\sum_{l=1}^{d}\lambda_{jl}t^l \quad (1\le j\le s)$$

without constant terms, and of degree at most d. Suppose that at least one of the ratios $\lambda_{jl}/\lambda_{km}$ is irrational. The number $\rho_{\boldsymbol{\phi}}(\tau,\nu)$ of solutions of

$$|\phi_1(x_1)+\ldots+\phi_s(x_s)-\nu|<\tau, \tag{1.9}$$

in positive integers, generalises the counter $\rho_{k,s}(\tau,\nu)$ in a natural way. Freeman's result is that there is a number $s_0(d)$, with $s_0(d)\sim 4d\log d$, and such that for fixed $\tau>0$ and large ν one has $\rho_{\boldsymbol{\phi}}(\tau,\nu)\ge 1$ (see Theorem 2 of [24]). Our investigation of (1.8) in §5.5 implies the following result, which we establish in chapter 7.

Theorem 1.10. *For any $d\in\mathbb{N}$, there exists a number $s_1(d)$ with $s_1(d)\sim 2d^2\log d$ and the following property. When $\phi_1,\ldots,\phi_s$ are polynomials of respective degrees $d_1,\ldots,d_s$ at most d subject to the conditions described in the preceding paragraph, then*

$$\rho_{\boldsymbol{\phi}}(\tau,\nu)=c(\boldsymbol{\phi})\tau\nu^{D-1}+o(\nu^{D-1}),$$

in which $D=d_1^{-1}+\ldots+d_s^{-1}$ and

$$c(\boldsymbol{\phi})=\frac{2\Gamma(1+d_1^{-1})\ldots\Gamma(1+d_s^{-1})}{\Gamma(D)\lambda_1^{1/d_1}\ldots\lambda_s^{1/d_s}}.$$

Under the current stipulations on $\boldsymbol{\phi}$, the inequality (1.9) implies an upper bound on x_j, whence the problem remains "definite". Freeman [24] also considers a cognate "indefinite" problem, and the corresponding analogue of Theorem 1.10 follows *mutatis mutandis*. These results are the first recorded instances of asymptotic formulae for $\rho_{\boldsymbol{\phi}}(\tau, \nu)$.

The individual chapters of this memoir are equipped with sections that describe the methods used herein in greater detail than would be possible at this point. The next chapter is intended as an introduction to the Davenport-Heilbronn method. Emphasis is on newer techniques that yield asymptotic formulae. In particular, we discuss a central contribution from the work of Bentkus and Götze [2] and of Freeman [21]. Rather than following their line of thought, we describe their device as an interference estimate for certain major arcs (Theorems 2.7 and 2.8). This yields an independent approach to asymptotic formulae for diophantine inequalities. In chapter 3, the work of chapter 2 is then illustrated with proofs for Theorems 1.1 and 1.5. A classical use of Plancherel's identity suffices here. In chapter 4, a discrete mean square approach is explained within a proof of Theorem 1.7. In chapter 5 we discuss the averaging tools from earlier papers in this series, that are then used to establish Theorems 1.2 and 1.6. Along the way, new mean value estimates for certain Weyl sums are obtained in §§5.4–5.5. The next chapter combines the ideas from chapter 5 with mean values of smooth Weyl sums, and also describes a lower bound method that was introduced in IV. The final chapter is an appendix on the problem described in Theorem 1.10.

The notation used in this memoir is standard, or otherwise explained at the appropriate stage of the proceedings. We write $e(\alpha) = \exp(2\pi i\alpha)$. The distance of a real number α to the nearest integer is $\|\alpha\|$. The integer part of α is $[\alpha]$, and $\lceil\alpha\rceil$ is the smallest integer n with $n \geq \alpha$. We apply the following convention concerning the letter ε. Whenever ε occurs in a statement, it is asserted that this statement is true for all real $\varepsilon > 0$, but constants implicit in Landau or Vinogradov symbols may depend on the actual value of ε.

2. The Fourier transform method

2.1. *Some classical integrals*

We begin with a review of the Fourier transform method, as pioneered by Davenport and Heilbronn. The scene is set up to cover more recent developments which yield asymptotic formulae, not only accidental lower bounds. Before we can embark on details, a few classical integral formulae are required that we now collect.

The Fourier transform of an integrable function $w : \mathbb{R} \to \mathbb{C}$ is

$$\widehat{w}(x) = \int_{-\infty}^{\infty} w(y)e(-xy)\,\mathrm{d}y, \tag{2.1}$$

and for any positive real number η, the functions

$$w_\eta(x) = \eta\Big(\frac{\sin \pi\eta x}{\pi\eta x}\Big)^2, \quad \widehat{w}_\eta(x) = \max\Big(0, 1 - \frac{|x|}{\eta}\Big) \tag{2.2}$$

are Fourier transforms of each other. Note that w and $\widehat{w}$ are non-negative. One can use (2.2) to construct a continuous approximation to the indicator function of an interval. When $\tau > 0$ and $\delta > 0$, define $W_{\tau,\delta} : \mathbb{R} \to [0,1]$ by

$$W_{\tau,\delta}(x) = \begin{cases} 1, & \text{for } |x| \le \tau, \\ 1 - (|x| - \tau)/\delta, & \text{for } \tau < |x| < \tau + \delta, \\ 0, & \text{for } |x| \ge \tau + \delta. \end{cases}$$

By (2.2), one has

$$W_{\tau,\delta}(x) = \Big(1 + \frac{\tau}{\delta}\Big)\widehat{w}_{\tau+\delta}(x) - \frac{\tau}{\delta}\widehat{w}_\tau(x). \tag{2.3}$$

Before we return to diophantine inequalities, we apply the function $w_\eta(x)$, given by (2.2), to define a measure $\mathrm{d}_\eta x$ on $\mathbb{R}$ with the property that for any bounded continuous function $\psi : \mathbb{R} \to \mathbb{C}$ and a measurable set $\mathcal{B}$, one has

$$\int_{\mathcal{B}} \psi(x)\,\mathrm{d}_\eta x = \int_{\mathcal{B}} \psi(x) w_\eta(x)\,\mathrm{d}x. \tag{2.4}$$

We omit explicit mention of the range of integration when $\mathcal{B} = \mathbb{R}$. This convenient notation avoids repeated occurences of the kernel w_η in most of the many integrals to follow.

2.2. *Counting solutions of diophantine inequalities*

The basic idea that underpins the strategy followed by Davenport and Heilbronn [17] is best explained in broad generality. Consider the polynomial $F \in \mathbb{Z}[x_1, \dots, x_s]$, and suppose that $u : \mathbb{Z}^s \to [0, \infty)$ is a weight that vanishes outside a finite subset of $\mathbb{Z}^s$. In practice u will be supported in a box depending on a size parameter, say B. For a positive real number τ, we then wish to evaluate the sum

$$\mathrm{P}(\tau; u) = \sum_{\substack{\mathbf{x} \in \mathbb{Z}^s \\ |F(\mathbf{x})| < \tau}} u(\mathbf{x}), \tag{2.5}$$

at least asymptotically, as $B \to \infty$. This is approached through the cognate yet analytically simpler expression

$$\mathrm{P}^*(\tau; u) = \sum_{\mathbf{x} \in \mathbb{Z}^s} u(\mathbf{x}) \widehat{w}_\tau(F(\mathbf{x})). \tag{2.6}$$

Occasionally, we shall use also the clumsier notation $\mathrm{P}_F(\tau; u)$, $\mathrm{P}^*_F(\tau; u)$ to point out the dependence on F more explicitly. If only a lower bound for $\mathrm{P}(\tau; u)$ is desired, then it will suffice to proceed through the inequality

$$\mathrm{P}(\tau; u) \geq \mathrm{P}^*(\tau; u) \tag{2.7}$$

that is readily inferred from (2.2), (2.5) and (2.6). Moreover, it is an exercise in elementary analysis to show that an asymptotic formula for $\mathrm{P}^*(\tau; u)$ implies a related one for the unweighted counter $\mathrm{P}(\tau; u)$. If one has to take care of error terms, this needs some dexterity. There are several ways to perform the transition. Freeman [22] and Wooley [44] describe a sandwich technique. Another option is to choose δ in the range $0 < \delta < \frac{1}{2}\tau$. Then observe that

$$\mathrm{P}(\tau; u) \leq \sum_{\mathbf{x} \in \mathbb{Z}^s} u(\mathbf{x}) W_{\tau,\delta}(F(\mathbf{x})), \tag{2.8}$$

an upper bound that uses only the definition of $W_{\tau,\delta}$ and the non-negativity of $u(\mathbf{x})$. However, when δ is much smaller than τ, the right hand side here should be approximately equal to $\mathrm{P}(\tau; u)$. In fact, the difference between the left and right hand sides of (2.8) arises only from solutions of $\tau < |F(\mathbf{x})| < \tau + \delta$, and by taking into account the actual definition of $W_{\tau,\delta}$, we find that

$$\mathrm{P}(\tau; u) \geq \sum_{\mathbf{x} \in \mathbb{Z}^s} u(\mathbf{x}) W_{\tau,\delta}(F(\mathbf{x})) - \mathrm{P}^*_{F-\tau}(\delta; u) - \mathrm{P}^*_{F+\tau}(\delta; u).$$

We may combine these inequalities and apply (2.3) to conclude as follows.

Lemma 2.1. *Let $0 < \delta < \frac{1}{2}\tau$. Then, in the notation introduced in this section, one has*

$$\mathrm{P}(\tau; u) = \Big(1 + \frac{\tau}{\delta}\Big) \mathrm{P}^*(\tau + \delta; u) - \frac{\tau}{\delta} \mathrm{P}^*(\tau; u) - E,$$

where E satisfies the inequalities $0 \leq E \leq \mathrm{P}^*_{F+\tau}(\delta; u) + \mathrm{P}^*_{F-\tau}(\delta; u)$.

2.3. *Weighted counting*

With the transition relations in (2.7) and Lemma 2.1 now in hand, we may concentrate on $\mathrm{P}^*(\tau;u)$. By (2.1), (2.2), (2.4) and (2.6) we have

$$\mathrm{P}^*(\tau;u) = \int \sum_{\mathbf{x}\in\mathbb{Z}^s} e(\alpha F(\mathbf{x}))u(\mathbf{x})\,\mathrm{d}_\tau\alpha. \tag{2.9}$$

It then suffices to establish an asymptotic formula for $\mathrm{P}^*(\tau;u)$; a mild uniformity in τ comes with it at no cost so that Lemma 2.1 yields the desired formula for $\mathrm{P}(\tau;u)$.

The integral (2.9) is the starting point for the Davenport-Heilbronn method. Before we enter this subject, we make the rather abstract discussion in §2.2 more concrete and show how this can be used to reduce the proofs of Theorems 1.1 and 1.5 to related weighted versions of these theorems.

We begin with additive cubic forms. The classical Weyl sum

$$f(\alpha) = \sum_{x\le X} e(\alpha x^3) \tag{2.10}$$

will be prominently featured. When $\lambda_1,\dots,\lambda_s$ are positive we choose

$$X = 2(\lambda_1^{-1/3}+\ldots+\lambda_s^{-1/3}+1)N^{1/3}. \tag{2.11}$$

Then, for any solution of $|\lambda_1x_1^3+\ldots+\lambda_sx_s^3-\nu|<1$ with $\nu\le N$, one has $x_j\le X$. Hence, we may take $F=\lambda_1x_1^3+\ldots+\lambda_sx_s^3-\nu$ and $u(\mathbf{x})$ as the indicator function on the box $1\le x_j\le X$ $(1\le j\le s)$. Then $\rho_s(\tau,\nu)=\mathrm{P}_F(\tau;u)$ in the notation of the previous section, and the weighted analogue $\mathrm{P}_F^*(\tau;u)$ now becomes

$$\rho_s^*(\tau,\nu) = \int f(\lambda_1\alpha)\ldots f(\lambda_s\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha,$$

as a special case of (2.9). In this form, Lemma 2.1 shows that whenever $0<\delta<\frac12\tau$, one has

$$\rho_s(\tau,\nu) = \Big(1+\frac{\tau}{\delta}\Big)\rho_s^*(\tau+\delta,\nu) - \frac{\tau}{\delta}\rho_s^*(\tau,\nu) + O\big(\rho_s^*(\delta,\nu+\tau)+\rho_s^*(\delta,\nu-\tau)\big). \tag{2.12}$$

For Theorem 1.2, we will have to work directly from this formula; see the last part of §5.2. In order to establish Theorem 1.1, on the other hand, we take $s=4$, and invoke the following weighted variant which involves the moment

$$\Upsilon(\tau,N) = \int_0^N \left|\rho_4^*(\tau,\nu) - \frac{\Gamma(\frac43)^3\tau\nu^{1/3}}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/3}}\right|^2 \mathrm{d}\nu. \tag{2.13}$$

Theorem 2.2. *Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be positive real numbers with λ_1/λ_2 irrational and algebraic. Then, uniformly in $(\log N)^{-3} < \tau \le 1$, one has*

$$\Upsilon(\tau, N) \ll \tau N^{5/3}(\log N)^{\varepsilon-3}.$$

This will be proved in §§3.1–3.3. For the time being, we take Theorem 2.2 for granted and deduce Theorem 1.1. One uses (2.12) together with the identity

$$2\tau = \Big(1 + \frac{\tau}{\delta}\Big)(\tau + \delta) - \frac{\tau^2}{\delta} - \delta \tag{2.14}$$

within the integral in Theorem 1.1. Then, by (2.13) and the trivial inequality $|\alpha + \beta|^2 \le 2(|\alpha|^2 + |\beta|^2)$, one finds that

$$\int_0^N \left|\rho_4(\tau, \nu) - \frac{2\Gamma(\frac{4}{3})^3 \tau \nu^{1/3}}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/3}}\right|^2 \mathrm{d}\nu$$
$$\ll \Big(\frac{\tau}{\delta}\Big)^2 \big(\Upsilon(\tau + \delta, N) + \Upsilon(\tau, N)\big) + \int_0^{2N} |\rho_4^*(\delta, \nu)|^2 \,\mathrm{d}\nu + \delta^2 N^{5/3}$$

holds uniformly for $0 < \delta < \frac{1}{2}\tau \le \frac{1}{2}$. The term $N^{5/3}\delta^2$ arises from integrating the term $-\delta$ in the expansion of 2τ, and the integral on the right hand side stems from the error terms $\rho^*(\delta, \nu \pm \tau)$ in (2.12). A similar argument gives

$$\int_0^{2N} |\rho_4^*(\delta, \nu)|^2 \,\mathrm{d}\nu \ll \Upsilon(\delta, 2N) + \int_0^{2N} \nu^{2/3}\delta^2 \,\mathrm{d}\nu,$$

and thus we conclude that

$$\int_0^N \left|\rho_4(\tau, \nu) - \frac{2\Gamma(\frac{4}{3})^3 \tau \nu^{1/3}}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/3}}\right|^2 \mathrm{d}\nu$$
$$\ll \Big(\frac{\tau}{\delta}\Big)^2 \big(\Upsilon(\tau + \delta, N) + \Upsilon(\tau, N)\big) + \Upsilon(\delta, 2N) + \delta^2 N^{5/3}.$$

The conclusion of Theorem 1.1 now follows from Theorem 2.2 on taking $\delta = \frac{1}{2}\tau^{3/4}(\log N)^{-3/4}$.

Similar techniques yield Theorem 1.5, but the details are more involved. For $\lambda_1, \lambda_2 > 0$ we study the linear problem (1.5). In this new context, put

$$X = 2(\lambda_1^{-1} + \lambda_2^{-1} + 1)N, \tag{2.15}$$

and observe that for any $\nu \le N$ and any solution of (1.5), one has $p_j \le X$. In (2.5) we insert $F = \lambda_1 x_1 + \lambda_2 x_2 - \nu$, and set $u(x_1, x_2) = 0$ unless x_1, x_2 are primes p_1, p_2 not exceeding X, in which case we take $u(p_1, p_2) = (\log p_1)(\log p_2)$. In the notation of Theorem 1.5 and that used in §2.2, we see that $\sigma(\tau,\nu) = \mathrm{P}_F(\tau;u)$. By (2.6) and (2.9), the formulae for $\mathrm{P}^*(\tau;u)$ mutate into

$$\sigma^*(\tau,\nu) = \int h(\lambda_1\alpha)h(\lambda_2\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha,$$

where now

$$h(\alpha) = \sum_{p\le X}(\log p)e(\alpha p)$$

is a Weyl sum over primes. The formula (2.12) remains valid, with ρ_s, ρ_s^* replaced by σ, σ^*, respectively. We conclude as follows.

Theorem 2.3. *With the hypotheses of Theorem 1.5, for any $A \ge 1$, there is a measurable function $E_\tau(\nu)$ such that, for $1 \le \nu \le N$, one has*

$$\sigma^*(\tau,\nu) = \frac{\tau\nu}{\lambda_1\lambda_2} + E_\tau(\nu) + O(\tau N(\log N)^{-A})$$

and

$$\int_0^N |E_\tau(\nu)|^2\,\mathrm{d}\nu \ll \tau N^{8/3+\varepsilon}.$$

We defer the proof of this result to §3.4, for now is the moment to deduce Theorem 1.5. Observing that the desired conclusion is trivial when τ is smaller than $N^{-1/3}$, we are entitled to assume the contrary. Equipped with the newly interpreted versions of (2.12) and (2.13), with σ in place of ρ_s, we deduce from (2.14) that for $0 < \delta < \frac{1}{2}\tau$, one has

$$\sigma(\tau,\nu) - \frac{2\tau\nu}{\lambda_1\lambda_2} = \Big(1+\frac{\tau}{\delta}\Big)E_{\tau+\delta}(\nu) - \frac{\tau}{\delta}E_\tau(\nu) + \frac{\delta\nu}{\lambda_1\lambda_2}$$
$$+ O\Big(\sigma^*(\delta,\nu+\tau) + \sigma^*(\delta,\nu-\tau) + \frac{\tau^2}{\delta}N(\log N)^{-8A}\Big).$$

Choosing $\delta = \tau(\log N)^{-4A}$, it is an easy exercise in the theory of uniform distribution to show that

$$(\log N)^{-2}\sigma^*(\delta,\nu) \le \mathrm{card}\,\{n,m\le X:\ |\lambda_1 n + \lambda_2 m - \nu| < \delta\} \ll \delta N.$$

Here it is worth recalling that $X \asymp N$ and λ_1/λ_2 is algebraic, and thus one even obtains an asymptotic formula for the counting problem in the middle term. The formula central to our discussion now reduces to

$$\Big|\sigma(\tau,\nu) - \frac{2\tau\nu}{\lambda_1\lambda_2}\Big| \ll (\log N)^{4A}(|E_{\tau+\delta}(\nu)| + |E_\tau(\nu)|) + \frac{\tau N}{(\log N)^{2A}}. \quad (2.16)$$

Next, consider any real number $\nu \in [\frac{1}{2}N, N]$ for which the inequality (1.6) holds. Then, by (2.16), one of the two inequalities

$$|E_\tau(\nu)| \geq \frac{\tau N}{(\log N)^{6A}} \quad \text{or} \quad |E_{\tau+\delta}(\nu)| \geq \frac{\tau N}{(\log N)^{6A}}$$

must also hold. However, by Theorem 2.3, the measure of all $\nu \leq N$, for which this lower bound for $|E_\tau(\nu)|$ holds, cannot exceed

$$\frac{(\log N)^{12A}}{\tau^2 N^2} \int_0^N |E_\tau(\nu)|^2 \,\mathrm{d}\nu \ll \tau^{-1} N^{2/3+\varepsilon}.$$

Since the same argument may be applied to $E_{\tau+\delta}(\nu)$, the conclusion of Theorem 1.5 follows via a dyadic dissection argument.

2.4. *The central interval*

As remarked earlier, the Davenport-Heilbronn method for diophantine inequalities embarks from (2.9). Whenever it succeeds, an asymptotic formula is produced where the main term arises from an interval centered at the origin, hereafter called the *central interval*. It has become common to refer to the latter interval as the major arc, by analogy with the circle method, but we reserve the term "major arcs" for classical major arcs.

We proceed in moderate detail and discuss the central interval for the integrals $\rho_s^*(\tau, \nu)$ and $\sigma^*(\tau, \nu)$. Our treatment of $\rho_s^*(\tau, \nu)$ can be taken as a model for any other application of the Fourier transform method to definite diophantine inequalities. There is a certain overlap with the exposition in Wooley [44], but the discussion there emphasises indefinite forms, and uses a different kernology. We shall conclude as follows.

Lemma 2.4. *Suppose that $s \geq 4$, that $\lambda_1, \ldots, \lambda_s$ are positive real numbers and that X is defined by (2.11). Let $C > 0$ denote a real number with $6C\lambda_j < 1$ for all $1 \leq j \leq s$, and let $\mathfrak{C} = [-CX^{-2}, CX^{-2}]$. In addition, put*

$$I = \int_{\mathfrak{C}} f(\lambda_1 \alpha) \ldots f(\lambda_s \alpha) e(-\alpha\nu) \,\mathrm{d}_\tau \alpha.$$

Then, uniformly in $0 < \tau \leq 1$ and $1 \leq \nu \leq N$, one has

$$I = \Gamma(\tfrac{4}{3})^s \Gamma(\tfrac{s}{3})^{-1} (\lambda_1 \cdots \lambda_s)^{-1/3} \tau \nu^{s/3-1} + O\big(1 + \tau X^{s-10/3}\big).$$

Proof. Let

$$v(\alpha) = \int_0^X e(\alpha\beta^3) \,\mathrm{d}\beta. \tag{2.17}$$

Then, according to Theorem 4.1 of Vaughan [35], in the range $|\alpha| \le \frac{1}{6}X^{-2}$ one has $f(\alpha) = v(\alpha) + O(1)$. Also, Theorem 7.3 of Vaughan [35] yields

$$v(\alpha) \ll X(1 + X^3|\alpha|)^{-1/3}. \tag{2.18}$$

For $\alpha \in \mathfrak{C}$, we therefore deduce that

$$f(\lambda_1\alpha)\dots f(\lambda_s\alpha) = v(\lambda_1\alpha)\dots v(\lambda_s\alpha) + O\big(1 + X^{s-1}(1 + X^3|\alpha|)^{-(s-1)/3}\big).$$

Then, on multiplying the previous display by $w_\tau(\alpha)e(-\alpha\nu)$ and integrating, we obtain

$$I = \int_{\mathfrak{C}} v(\lambda_1\alpha)\dots v(\lambda_s\alpha)e(-\alpha\nu)\,\mathrm{d}_\tau\alpha + O(1 + \tau X^{s-4}\log X).$$

By (2.18), the *singular integral*

$$I_\infty = \int v(\lambda_1\alpha)\dots v(\lambda_s\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha$$

converges, and by (2.18) and (2.2), for any $Y > 0$, one has

$$\int_Y^\infty |v(\lambda_1\alpha)\dots v(\lambda_s\alpha)|\,\mathrm{d}_\tau\alpha \ll \tau\int_Y^\infty \alpha^{-s/3}\,\mathrm{d}\alpha.$$

It is now immediate that

$$I = I_\infty + O(1 + \tau X^{s-10/3}). \tag{2.19}$$

Within the singular integral, we resubstitute (2.17) and apply (2.1) and (2.4) to arrive at the interim identity

$$I_\infty = \int_{[0,X]^s} \widehat{w}_\tau(\lambda_1\beta_1^3 + \dots + \lambda_s\beta_s^3 - \nu)\,\mathrm{d}\boldsymbol{\beta}.$$

Since the λ_j are positive, it follows from (2.2) that we may extend the range of integration to $[0,\infty)^s$. A change of variable then yields the alternative formula

$$I_\infty = (\lambda_1\dots\lambda_s)^{-1/3}\int_{[0,\infty)^s} \widehat{w}_\tau(z_1^3 + \dots + z_s^3 - \nu)\,\mathrm{d}\mathbf{z}.$$

Consider the equation $t = z_1^3 + \dots + z_s^3$, which defines a surface in $\mathbb{R}^s$ of codimension 1. The area of this manifold in the quadrant with all z_j positive is $\Gamma(\frac{4}{3})^s\Gamma(\frac{s}{3})^{-1}t^{\frac{s}{3}-1}$. Hence, by the transformation formula and Fubini's theorem, one confirms that

$$I_\infty = (\lambda_1\dots\lambda_s)^{-1/3}\Gamma\Big(\frac{4}{3}\Big)^s\Gamma\Big(\frac{s}{3}\Big)^{-1}\int_0^\infty t^{s/3-1}\widehat{w}_\tau(t-\nu)\,\mathrm{d}t.$$

By (2.2), the remaining integral on the right hand side is equal to

$$\int_{-\tau}^{\tau}\Big(1-\frac{|\alpha|}{\tau}\Big)(\nu+\alpha)^{s/3-1}\,\mathrm{d}\alpha=\tau\nu^{s/3-1}+O(\tau^2\nu^{s/3-2}),$$

and the lemma follows from (2.19). □

The analysis of the prime variables case is deeper because we need a rather wide central interval, for a reason that will become more transparent in due course. In such cases, the distribution of primes in short intervals comes into play. Fortunately, our discussion may be abbreviated by appealing to Brüdern, Cook and Perelli [6].

Lemma 2.5. *Let* $\lambda_1,\lambda_2>0$ *and* $0<\tau<1$, *and suppose that* X *is defined by means of* (2.15). *In addition, let* $\mathfrak{C}=[-X^{-1/2},X^{-1/2}]$. *Then, for any* $A>1$, *and uniformly in* $0<\tau\le 1$ *and* $1\le\nu\le N$, *one has*

$$\int_{\mathfrak{C}}h(\lambda_1\alpha)h(\lambda_2\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha=\frac{\tau\nu}{\lambda_1\lambda_2}+O(\tau X(\log X)^{-A}).$$

Proof. Let

$$h^*(\alpha)=\sum_{x\le X}e(\alpha x).$$

Then one may apply the methods underlying the proof of Lemma 2 of [6] to establish the estimate

$$\int_{\mathfrak{C}}|h(\lambda_j\alpha)-h^*(\lambda_j\alpha)|^2\,\mathrm{d}\alpha\ll X(\log X)^{-2A}. \tag{2.20}$$

Here we note that although Lemma 2 of [6] states (2.20) only with $A=1$, an inspection of the proof, which combines only Lemma 1 and estimate (5) of that paper, shows that any positive value of A is permissible. Next, let

$$v_1(\alpha)=\int_0^X e(\alpha\beta)\,\mathrm{d}\beta.$$

By Euler's summation formula, one finds that $h^*(\alpha)-v_1(\alpha)\ll 1+X|\alpha|$, whence from (2.20) we obtain

$$\int_{\mathfrak{C}}|h(\lambda_j\alpha)-v_1(\lambda_j\alpha)|^2\,\mathrm{d}\alpha\ll X(\log X)^{-2A}.$$

Then, on noting the trivial estimate $w_\tau(\alpha)\ll\tau$ evident from (2.2), it follows from Schwarz's inequality that

$$\begin{aligned}&\int_{\mathfrak{C}}h(\lambda_1\alpha)h(\lambda_2\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha\\&=\int_{\mathfrak{C}}v_1(\lambda_1\alpha)v_1(\lambda_2\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha+O(\tau X(\log X)^{-A}).\end{aligned} \tag{2.21}$$

We can now proceed as we have explained in detail for sums of cubes. By partial integration, one has $v_1(\alpha) \ll X(1+X|\alpha|)^{-1}$, and consequently, the integral on the right hand side of (2.21) can be extended to the whole real line, with the introduction of acceptable errors. Then, applying Fourier inversion as before we confirm that

$$\int_{-\infty}^{\infty} v_1(\lambda_1\alpha)v_1(\lambda_2\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha = \frac{\tau\nu}{\lambda_1\lambda_2},$$

the details being considerably simpler. This proves the lemma. □

2.5. *The interference principle*

In the previous section, we evaluated the contribution from the central interval $\mathfrak{C}$ to the Fourier transform that counts solutions of a diophantine inequality. The complementary set $\mathfrak{c} = \mathbb{R}\backslash\mathfrak{C}$ consists of two disjoint half-lines, and we therefore refer to it as the *complementary compositum.* For a successful analysis, its contribution to the count should be of a lower order of magnitude. The most important ingredient in any proof of this is an *interference principle* asserting that, when λ_1, λ_2 are non-zero real numbers, and λ_1/λ_2 is irrational, then two exponential sums such as $f(\lambda_1\alpha)$ and $f(\lambda_2\alpha)$, or $h(\lambda_1\alpha)$ and $h(\lambda_2\alpha)$, say, cannot be large simultaneously unless α lies in the central interval. However, loosely speaking, when $|f(\lambda_j\alpha)|$ is large, then as a consequence of Weyl's inequality, or a suitable variant thereof, one finds that $\lambda_j\alpha$ has a rational approximation a_j/q_j with small denominator. If this happens simultaneously for $j=1,2$, then $a_1q_2/(a_2q_1)$ is an approximation to λ_1/λ_2, and so the measure of the set of all α where $|f(\lambda_1\alpha)|$ and $|f(\lambda_2\alpha)|$ are simultaneously large, should be quite small. In this form, the interference principle becomes a statement about diophantine approximations alone, and references to exponential sums can be removed entirely. For another, rather different view of this phenomenon, see section 3 of Brüdern [5].

We shall present here a simple derivation of the interference principle along the lines indicated, based on ideas of Watson [38]. It appears to us that the potential of this approach has been overlooked in the past. As we shall demonstrate below, our method provides easy access to asymptotic formulae for diophantine inequalities, avoiding to a large extent the entangled interplay between diophantine approximations and major arc information for exponential sums, as in the celebrated works of Bentkus and Götze [2], and Freeman [21], who were the first to obtain such asymptotic formulae at all. The stronger bounds that are available when λ_1/λ_2 is not only irrational but also algebraic, moreover, follow from the same principles.

Before making these comments precise, we need to introduce some notation. Let $N \geq 1$ denote the main parameter. For $1 \leq Q \leq \frac{1}{2}\sqrt{N}$ the intervals $|q\alpha - a| \leq Q/N$ with $1 \leq q \leq Q$, $a \in \mathbb{Z}$ and $(a, q) = 1$, are pairwise disjoint. Their union, the *major arcs* $\mathfrak{M}(Q)$, forms a 1-periodic set. The subset $\mathfrak{N}(Q) = \mathfrak{M}(Q) \cap [0, 1]$ is the familiar set of major arcs in the classical circle method. We also define here the *minor arcs*

$$\mathfrak{m}(Q) = \mathbb{R}\backslash\mathfrak{M}(Q), \quad \mathfrak{n}(Q) = [0, 1]\backslash\mathfrak{N}(Q),$$

although these will not be needed until the next section. Watson's method is imported through Lemma 4 of Brüdern, Cook and Perelli [6] that we restate as Lemma 2.6. Temporarily, we suppose only that λ_1, λ_2 are *non-zero* real numbers, so that our main estimates Theorems 2.7 and 2.8 apply also to indefinite problems. With $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$ fixed, we define

$$\mathfrak{K}(Q_1, Q_2) = \{\alpha \in \mathbb{R} : \lambda_j\alpha \in \mathfrak{M}(Q_j) \ (j = 1, 2)\}.$$

In addition, when $y > 0$, we put

$$\mathfrak{K}_y(Q_1, Q_2) = \{\alpha \in \mathfrak{K}(Q_1, Q_2) : y < |\alpha| \leq 2y\}.$$

Lemma 2.6. *Let λ_1, λ_2 be non-zero real numbers such that λ_1/λ_2 is irrational. There exists a positive real number $\varepsilon_0 = \varepsilon_0(\lambda_1, \lambda_2)$ with the following property. Suppose that $1 \leq Q_j \leq \frac{1}{2}\sqrt{N}$ $(j = 1, 2)$, and $r \in \mathbb{N}$ satisfies $r \leq \varepsilon_0 N/(Q_1Q_2)$ and $\|r\lambda_1/\lambda_2\| < 1/r$. Then, for any $y > 0$ with $|\lambda_j|y \geq 2Q_j/N$ $(j = 1, 2)$, one has* $\operatorname{mes}\mathfrak{K}_y(Q_1, Q_2) \ll yN^{-1}Q_1Q_2r^{-1}$.

Note that $\mathfrak{N}(Q)$ has measure about Q^2N^{-1}. An application of Schwarz's inequality therefore reveals that $\operatorname{mes}\mathfrak{K}_y(Q_1, Q_2) \ll yN^{-1}Q_1Q_2$. Lemma 2.6 improves on this estimate by a factor $1/r$, and it is this saving that implies that $\lambda_1\alpha$ and $\lambda_2\alpha$ simultaneously lie on major arcs only for a slim set of real numbers α.

For a non-zero real number λ, define

$$T_\lambda(R) = \max\{r \in \mathbb{N} : r \leq R, \|\lambda r\| \leq 1/r\}. \tag{2.22}$$

When λ is irrational, then $\|\lambda r\| \leq 1/r$ has infinitely many solutions, and consequently $T_\lambda(R) \to \infty$ as $R \to \infty$. If r_m is the sequence of solutions of $\|\lambda r\| \leq 1/r$, arranged in increasing order, then for algebraic irrational numbers λ, Roth's theorem gives $r_{m+1} \ll r_m^{1+\varepsilon}$. Therefore, in this case,

$$T_\lambda(R) \gg R^{1-\varepsilon}. \tag{2.23}$$

We can now put Lemma 2.6 into a form more readily applied. Subject to the conditions of this lemma, we deduce from (2.2) and (2.4) that

$$\int_{\mathfrak{K}_y(Q_1,Q_2)} \mathrm{d}_\tau\alpha \ll Q_1Q_2(NT)^{-1}\min(\tau y,(\tau y)^{-1}),$$

where $T = T_{\lambda_1/\lambda_2}(\varepsilon_0 N/(Q_1Q_2))$. We choose a number Y with $|\lambda_j|Y \geq 2Q_j/N$ $(j=1,2)$ and sum the previous estimate over $y = 2^l Y$. This gives

$$\int\limits_{\substack{|\alpha|\geq Y\\ \alpha\in\mathfrak{K}(Q_1,Q_2)}} \mathrm{d}_\tau\alpha \ll Q_1Q_2(NT)^{-1}. \tag{2.24}$$

Theorem 2.7. *Let λ_1,λ_2 be non-zero real numbers. If λ_1/λ_2 is irrational and algebraic, then uniformly in $0<\tau\leq 1$, $Q_j \leq \frac{1}{2}\sqrt{N}$ and $Y \geq 2Q_j/(|\lambda_j|N)$ $(j=1,2)$, one has*

$$\int\limits_{\substack{|\alpha|\geq Y\\ \alpha\in\mathfrak{K}(Q_1,Q_2)}} \mathrm{d}_\tau\alpha \ll N^{\varepsilon-2}Q_1^2Q_2^2.$$

The proof is immediate from (2.23) and (2.24).

Theorem 2.8. *Let $Q=Q(N)$ be a function that is increasing, unbounded, and satisfies $Q(N)/\sqrt{N}\to 0$ as $N\to\infty$. Let λ_1,λ_2 be non-zero real numbers such that λ_1/λ_2 is irrational. Then, uniformly in $0<\tau\leq 1$, $1\leq Q_j\leq Q(N)$ and $Y\geq 2Q(N)/(|\lambda_j|N)$, one has*

$$\int\limits_{\substack{|\alpha|\geq Y\\ \alpha\in\mathfrak{K}(Q_1,Q_2)}} \mathrm{d}_\tau\alpha \ll N^{-1}Q_1Q_2T_{\lambda_1/\lambda_2}\Big(\frac{\varepsilon_0 N}{Q(N)^2}\Big)^{-1}.$$

Again, this is merely a restatement of (2.24) if one observes that under the current hypotheses, one has $Q_1Q_2\leq Q(N)^2$, and $T_\lambda(R)$ is a non-decreasing function. We note that the condition $N/Q(N)^2\to\infty$ ensures that the upper bound in Theorem 2.8 is $o(N^{-1}Q_1Q_2)$ as $N\to\infty$.

3. Classical mean square methods

3.1. *Plancherel's identity*

In this chapter, we complete the proofs of Theorems 1.1 and 1.5 by demonstrating Theorems 2.2 and 2.3. However, our primary concern is to illustrate, in this and related enterprises, the use of Plancherel's identity for take-off and Wooley's amplifier [44] coupled with the interference principle for landing.

As in previous sections, we begin with additive cubic forms. Under the hypotheses of Theorem 2.2, for any measurable set $\mathfrak{A} \subset \mathbb{R}$ we write

$$\rho^*_{\mathfrak{A}}(\tau, \nu) = \int_{\mathfrak{A}} f(\lambda_1\alpha)f(\lambda_2\alpha)f(\lambda_3\alpha)f(\lambda_4\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha. \tag{3.1}$$

We define the central interval $\mathfrak{C}$ as in Lemma 2.4, and the complementary compositum $\mathfrak{c}$ as in §2.5, and then have $\rho^*_4(\tau,\nu) = \rho^*_{\mathfrak{C}}(\tau,\nu) + \rho^*_{\mathfrak{c}}(\tau,\nu)$. Consequently, recalling (2.13), we find that

$$\Upsilon(\tau, N) \ll \int_0^N \left| \rho^*_{\mathfrak{C}}(\tau,\nu) - \frac{\Gamma(\frac{4}{3})^3\tau\nu^{1/3}}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/3}} \right|^2 \mathrm{d}\nu + \int_0^N |\rho^*_{\mathfrak{c}}(\tau,\nu)|^2\,\mathrm{d}\nu.$$

Next, applying Lemma 2.4 to estimate the term involving the central interval, we deduce that

$$\Upsilon(\tau, N) \ll N(1 + N^{4/9}\tau^2) + \int_0^N |\rho^*_{\mathfrak{c}}(\tau,\nu)|^2\,\mathrm{d}\nu. \tag{3.2}$$

Now, by (3.1) and (2.4), the number $\rho_{\mathfrak{c}}(\tau,\nu)$ is the Fourier transform, at ν, of the function that is

$$f(\lambda_1\alpha)f(\lambda_2\alpha)f(\lambda_3\alpha)f(\lambda_4\alpha)w_\tau(\alpha)$$

for $\alpha \in \mathfrak{c}$, and 0 on $\mathfrak{C}$. By Plancherel's identity,

$$\int_{-\infty}^{\infty} |\rho^*_{\mathfrak{c}}(\tau,\nu)|^2\,\mathrm{d}\nu = \int_{\mathfrak{c}} |f(\lambda_1\alpha)\ldots f(\lambda_4\alpha)|^2 w_\tau(\alpha)^2\,\mathrm{d}\alpha,$$

from which we deduce, via (2.2) and (2.4), the important inequality

$$\int_0^N |\rho^*_{\mathfrak{c}}(\tau,\nu)|^2\,\mathrm{d}\nu \le \tau \int_{\mathfrak{c}} |f(\lambda_1\alpha)\ldots f(\lambda_4\alpha)|^2\,\mathrm{d}_\tau\alpha \tag{3.3}$$

through which the estimation will proceed.

3.2. *Some mean values*

We summarize here a few standard bounds for Weyl sums $f(\alpha)$. It will be convenient, temporarily, to write $\mathfrak{M} = \mathfrak{M}(X^{3/4})$ and $\mathfrak{m} = \mathfrak{m}(X^{3/4})$, with $\mathfrak{N}$ and $\mathfrak{n}$ defined mutatis mutandis. Combining the methods used to prove Lemma 1 in Vaughan [33] with the bounds for Hooley's Δ-function in Hall and Tenenbaum [26], we obtain an enhanced form of Weyl's inequality which asserts that

$$\sup_{\alpha\in\mathfrak{m}} |f(\alpha)| \ll X^{3/4}(\log X)^{1/4+\varepsilon}. \tag{3.4}$$

Boklan [3] supplies the bound

$$\int_{\mathfrak{n}} |f(\alpha)|^8 \, \mathrm{d}\alpha \ll X^5 (\log X)^{\varepsilon - 3} \tag{3.5}$$

as an improvement over Theorem B of Vaughan [33]. Standard applications of the circle method (see Chapter 4 of [35]) yield the complementary estimates

$$\int_{\mathfrak{N}} |f(\alpha)|^4 \, \mathrm{d}\alpha \ll X^{1+\varepsilon}, \quad \int_{\mathfrak{N}} |f(\alpha)|^{4+\theta} \, \mathrm{d}\alpha \ll X^{1+\theta}, \tag{3.6}$$

the latter being valid for any fixed $\theta > 0$. This last bound combines with (3.5) to deliver the estimate

$$\int_0^1 |f(\alpha)|^8 \, \mathrm{d}\alpha \ll X^5, \tag{3.7}$$

a conclusion that is also implied by Theorem 2 of Vaughan [33]. We now transport these mean value bounds into integrals over 1-periodic sets, against the measure $\mathrm{d}_\tau \alpha$.

Lemma 3.1. *Let $G : \mathbb{R} \to \mathbb{C}$ be a function of period 1 that is integrable on $[0, 1]$. Then, for any $\tau > 0$ and any $u \in \mathbb{R}$, one has*

$$\int G(\alpha) e(-\alpha u) \, \mathrm{d}_\tau \alpha = \sum_{n=-\infty}^{\infty} \widehat{w}_\tau(n-u) \int_0^1 G(\alpha) e(-\alpha n) \, \mathrm{d}\alpha.$$

Proof. This is a special case of formula (4) in Brüdern [4], with $w_\tau(\alpha)$ in the role of the kernel K employed in [4]. Note that the sum on the right is over $|n - u| \le \tau$ only. □

As an example, take $u = 0$, and put $G(\alpha) = |f(\alpha)|^8$ when $\alpha \in \mathfrak{m}$, and $G(\alpha) = 0$ otherwise. Then

$$\int_{\mathfrak{m}} |f(\alpha)|^8 \, \mathrm{d}_\tau \alpha = \sum_{|n| \le \tau} \widehat{w}_\tau(n) \int_{\mathfrak{n}} |f(\alpha)|^8 e(-\alpha n) \, \mathrm{d}\alpha,$$

whence, in particular,

$$\int_{\mathfrak{m}} |f(\alpha)|^8 \, \mathrm{d}_\tau \alpha \ll X^5 (\log X)^{\varepsilon - 3}. \tag{3.8}$$

Similarly, when $\tau \ll 1$ and $\theta > 0$, one infers from (3.6) that

$$\int_{\mathfrak{M}} |f(\alpha)|^4 \, \mathrm{d}_\tau \alpha \ll X^{1+\varepsilon}, \quad \int_{\mathfrak{M}} |f(\alpha)|^{4+\theta} \, \mathrm{d}_\tau \alpha \ll X^{1+\theta}, \tag{3.9}$$

and from (3.7) that

$$\int |f(\alpha)|^8 \, \mathrm{d}_\tau \alpha \ll X^5. \tag{3.10}$$

3.3. *The amplification technique*

The first steps in the estimation of the right hand side of (3.3) follow Wooley [44]. We need to cover $\mathfrak{c}$ by the sets

$$\begin{aligned} \mathfrak{d} &= \{\alpha \in \mathfrak{c} : \ \lambda_j \alpha \in \mathfrak{m} \ (1 \le j \le 4)\}, \\ \mathfrak{D} &= \{\alpha \in \mathfrak{c} : \ \lambda_j \alpha \in \mathfrak{M} \ (1 \le j \le 4)\}, \\ \mathfrak{E}_{ij} &= \{\alpha \in \mathfrak{c} : \ \lambda_i \alpha \in \mathfrak{M}, \ \lambda_j \alpha \in \mathfrak{m}\} \qquad (1 \le i, j \le 4, \ i \ne j). \end{aligned}$$

Each of these sets $\mathfrak{a}$ requires a different argument. For convenience, we write

$$I(\mathfrak{a}) = \int_{\mathfrak{a}} |f(\lambda_1 \alpha) \ldots f(\lambda_4 \alpha)|^2 \, \mathrm{d}_\tau \alpha.$$

By Hölder's inequality, one infers that

$$I(\mathfrak{d}) \le \prod_{j=1}^{4} \Big(\int_{\lambda_j \alpha \in \mathfrak{m}} |f(\lambda_j \alpha)|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/4}.$$

Since a change of variable reveals that

$$\int_{\lambda \alpha \in \mathfrak{m}} |f(\lambda \alpha)|^8 \, \mathrm{d}_\tau \alpha = \int_{\mathfrak{m}} |f(\alpha)|^8 \, \mathrm{d}_{\tau/\lambda} \alpha,$$

one derives from (3.8) the important bound

$$I(\mathfrak{d}) \ll X^5 (\log X)^{\varepsilon - 3}. \tag{3.11}$$

Next, we use (3.4) and Hölder's inequality to conclude that

$$\begin{aligned} I(\mathfrak{E}_{12}) \ll X^{3/2+\varepsilon} & \Big(\int_{\lambda_1 \alpha \in \mathfrak{M}} |f(\lambda_1 \alpha)|^4 \, \mathrm{d}_\tau \alpha \Big)^{1/2} \\ & \times \Big(\int |f(\lambda_3 \alpha)|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/4} \Big(\int |f(\lambda_4 \alpha)|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/4}. \end{aligned}$$

Thus, by changes of variable together with (3.9) and (3.10), it follows that $I(\mathfrak{E}_{12}) \ll X^{9/2+\varepsilon}$. By symmetry, this shows that

$$I(\mathfrak{E}_{ij}) \ll X^{9/2+\varepsilon} \quad (1 \le i, j \le 4, \ i \ne j). \tag{3.12}$$

The *amplification* is now complete: since $\mathfrak{c}$ is the union of $\mathfrak{d}, \mathfrak{D}$ and the $\mathfrak{E}_{ij}$, then in view of (3.11) and (3.12), it suffices to consider $\mathfrak{D}$. Here all $\lambda_j \alpha$ lie on a major arc. But by comparison with the analysis just undertaken, major arc moments are far easier to control (compare (3.5), (3.6)), and the interference principle can be brought into play.

For $|q\alpha - a| \le QX^{-3}$ and $q \le Q \le X^{3/4}$ it follows from Theorem 4.1 and Lemma 4.6 of [35] that

$$f(\alpha) \ll X(q + X^3|q\alpha - a|)^{-1/3}, \tag{3.13}$$

and consequently, when $\alpha \in \mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)$, we find that $f(\alpha) \ll XQ^{-1/3}$. Therefore, slicing $\mathfrak{D}$ into sections

$$\mathfrak{D}(Q_1, Q_2) = \{\alpha \in \mathfrak{D} :\ \lambda_j\alpha \in \mathfrak{M}(2Q_j)\backslash\mathfrak{M}(Q_j) \text{ for } j = 1, 2\}, \tag{3.14}$$

with $1 \le Q_j \le X^{3/4}$, we see from Theorem 2.7 that

$$\int_{\mathfrak{D}(Q_1,Q_2)} |f(\lambda_1\alpha)f(\lambda_2\alpha)|^2 \,\mathrm{d}_\tau\alpha \ll N^{\varepsilon-2}X^4(Q_1Q_2)^{4/3} \ll N^{\varepsilon}.$$

By a dyadic dissection argument and trivial bounds for $|f(\lambda_3\alpha)f(\lambda_4\alpha)|$, it therefore follows that $I(\mathfrak{D}) \ll N^{\varepsilon}X^4$. In combination with (3.11) and (3.12), we may infer that the integral on the left hand side of (3.3) is $O(\tau X^5(\log X)^{\varepsilon-3})$, and Theorem 2.2 follows from (3.2). In view of the discussion following the statement of Theorem 2.2, this also completes the proof of Theorem 1.1.

3.4. *Linear forms in primes*

We now establish Theorem 2.3 by an argument paralleling that of the previous two sections. By Lemma 2.5, if we put $\mathfrak{c} = \{\alpha : |\alpha| \ge X^{-1/2}\}$ and

$$E_\tau(\nu) = \int_{\mathfrak{c}} h(\lambda_1\alpha)h(\lambda_2\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha, \tag{3.15}$$

then just as in the discussion leading to (3.3), Plancherel's identity gives

$$\int_0^N |E_\tau(\nu)|^2 \,\mathrm{d}\nu \ll \tau \int_{\mathfrak{c}} |h(\lambda_1\alpha)h(\lambda_2\alpha)|^2 \,\mathrm{d}_\tau\alpha. \tag{3.16}$$

The dissection of $\mathfrak{c}$ this time is much simpler. For $j = 1$ and 2, we consider

$$\mathfrak{m}_j = \{\alpha \in \mathfrak{c} :\ \lambda_j\alpha \in \mathfrak{m}(X^{1/3})\}.$$

Then $\mathfrak{c}$ is the union of $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{c} \cap \mathfrak{K}(X^{1/3}, X^{1/3})$.

Vinogradov's estimate for exponential sums (Vaughan [35], Theorem 3.1) shows both that

$$\sup_{\alpha\in\mathfrak{m}(X^{1/3})} |h(\alpha)| \ll X^{5/6}(\log X)^4,$$

and, whenever $Q \le X^{1/3}$, that

$$\sup_{\alpha \in \mathfrak{M}(2Q)\setminus\mathfrak{M}(Q)} |h(\alpha)| \ll XQ^{-1/2}(\log X)^4. \tag{3.17}$$

Paired with the mean square bound

$$\int |h(\lambda_1\alpha)|^2 \, \mathrm{d}_\tau\alpha = \sum_{\substack{\lambda_1|p_1-p_2|<\tau \\ p_1,p_2\le X}} (\log p_1)(\log p_2) \ll X\log X, \tag{3.18}$$

the first of these estimates yields

$$\int_{\mathfrak{m}_2} |h(\lambda_1\alpha)h(\lambda_2\alpha)|^2 \, \mathrm{d}_\tau\alpha \ll X^{8/3}(\log X)^9,$$

and the same is true for the contribution from $\mathfrak{m}_1$, by symmetry. For the set $\mathfrak{K}(X^{1/3}, X^{1/3})$, we apply the same slicing technique as we used for $\mathfrak{D}$ in §3.3. Then, importing (3.17) into Theorem 2.7, we find that

$$\int_{\mathfrak{c}\cap\mathfrak{K}(X^{1/3},X^{1/3})} |h(\lambda_1\alpha)h(\lambda_2\alpha)|^2 \, \mathrm{d}_\tau\alpha \ll X^{8/3+\varepsilon},$$

an estimate that may also be found on p. 97 of [6]. Collecting the upper bounds obtained thus far, we conclude that

$$\int_{\mathfrak{c}} |h(\lambda_1\alpha)h(\lambda_2\alpha)|^2 \, \mathrm{d}_\tau\alpha \ll X^{8/3+\varepsilon}, \tag{3.19}$$

and Theorem 2.3 follows from (3.16) and Lemma 2.5.

3.5. *Bessel's inequality*

The continuous averages in Theorems 1.1, 2.2 and 2.3 could be replaced by discrete ones. In such a setting, one chooses an increasing sequence of test points ν_m that is roughly of linear growth; see [6] for one of the many possibilities to make this precise. One would then study, for example, a mean square of the type

$$\sum_{\nu_m \le N} \left| \rho_4(\tau, \nu_m) - \frac{2\Gamma(\frac{4}{3})^3 \tau \nu_m^{1/3}}{(\lambda_1\lambda_2\lambda_3\lambda_4)^{1/3}} \right|^2$$

as the appropriate analogue of the integral in Theorem 1.1. This approach, which has been used successfully by Brüdern, Cook and Perelli [6], Parsell [30], and others, is roughly of the same strength as the methods described herein. Bessel's inequality performs the averaging in the discrete world, and replaces Plancherel's identity in the work of §3.1 and elsewhere. For thinner averages, it appears that it is almost mandatory to work with discrete test points, and this will be the theme in most of the following sections.

4. Semi-classical averaging

4.1. *Another mean square approach*

The main purpose of this chapter is to establish Theorem 1.7. Thus, we continue the discussion begun in §§3.4 and 3.5, and examine the distribution of the linear form $\lambda_1 p_1 + \lambda_2 p_2$ near the sequence

$$\nu_m = \exp((\log m)^\gamma), \tag{4.1}$$

where $1 < \gamma < \frac{3}{2}$ is fixed once and for all. As much as is possible, the notation from the proof of Theorem 1.5 is kept throughout this chapter. The parameters X and N are linked as in (2.15), and we apply the same notation as in §3.4 and the statement of Lemma 2.5. This defines the central interval and its complement $\mathfrak{c}$. With $E_\tau(\nu)$ as in (3.15), we recall that Lemma 2.5 asserts that for any $A > 1$, uniformly in $0 < \tau \le 1$ and $1 \le \nu \le N$, one has

$$\sigma^*(\tau,\nu) = \frac{\tau\nu}{\lambda_1\lambda_2} + E_\tau(\nu) + O(\tau N(\log N)^{-A}). \tag{4.2}$$

Now define M through the equation

$$\exp((\log 2M)^\gamma) = N.$$

Then $\log M \asymp (\log N)^{1/\gamma}$, and whenever $m \le 2M$ one has $\nu_m \le N$.

Lemma 4.1. *Let $1 < \gamma < \frac{3}{2}$. Also, let $\lambda_1, \lambda_2 > 0$, and suppose that λ_1/λ_2 is an algebraic irrational. Then, there exists $\kappa > 0$ such that*

$$\sum_{M<m\le 2M} |E_\tau(\nu_m)|^2 \ll MN^2 \exp\big(-2\kappa(\log M)^{3-2\gamma}\big).$$

This implies Theorem 1.7, for in view of (4.2) we infer that for any fixed $0 < \tau < 1$, the asymptotic formula

$$\sigma^*(\tau,\nu) = \frac{\tau\nu}{\lambda_1\lambda_2} + O(N(\log N)^{-A})$$

holds for all but $O(M\exp(-\kappa(\log M)^{3-2\gamma}))$ of the values $\nu = \nu_m$ with $M < m \le 2M$. By the transfer principle (Lemma 2.1) and an argument almost identical to the one given after the statement of Theorem 2.3, this can be reformulated as an asymptotic formula for $\sigma(\tau,\nu)$. Theorem 1.7 then follows by summing over dyadic intervals.

4.2. *Exponential sums over test sequences*

Let

$$\Phi(\zeta) = \sum_{M<m\le 2M} e(\zeta\nu_m), \quad H(\zeta) = h(\lambda_1\zeta)h(\lambda_2\zeta). \tag{4.3}$$

Then, by (3.15), one has

$$\sum_{M<m\le 2M} |E_\tau(\nu_m)|^2 = \iint_{\mathfrak{c}\times\mathfrak{c}} H(\alpha)H(-\beta)\Phi(\beta-\alpha)\,\mathrm{d}_\tau\alpha\,\mathrm{d}_\tau\beta. \tag{4.4}$$

The next lemma is a special case of Theorem 2 of Brüdern and Perelli [15].

Lemma 4.2. *Let* $1<\gamma<\frac{3}{2}$ *and* $0<\delta<\min(\frac{1}{100},\gamma-1)$*. Then, there is a real number* $\kappa>0$ *such that uniformly in* $N^{\delta-1}\le|\zeta|\le N^\delta$*, one has*

$$\Phi(\zeta)\ll M\exp(-2\kappa(\log M)^{3-2\gamma}).$$

In order to establish Lemma 4.1, we split the integration in (4.4) into the three regions

$$\begin{aligned}
\mathfrak{T} &= \{(\alpha,\beta)\in\mathfrak{c}\times\mathfrak{c}:\ |\alpha-\beta|>2N^\delta\},\\
\mathfrak{U} &= \{(\alpha,\beta)\in\mathfrak{c}\times\mathfrak{c}:\ N^{\delta-1}<|\alpha-\beta|\le 2N^\delta\},\\
\mathfrak{V} &= \{(\alpha,\beta)\in\mathfrak{c}\times\mathfrak{c}:\ |\alpha-\beta|\le N^{\delta-1}\}.
\end{aligned}$$

The corresponding contributions to (4.4) are denoted by $J(\mathfrak{T})$, $J(\mathfrak{U})$, $J(\mathfrak{V})$ respectively, and we examine each in turn.

The set $\mathfrak{T}$ presents little difficulty. When $|\alpha-\beta|>2N^\delta$, then $\max(|\alpha|,|\beta|)>N^\delta$. By symmetry and a trivial bound for $\Phi(\zeta)$, this implies that

$$|J(\mathfrak{T})|\le 4M\int |H(\beta)|\,\mathrm{d}_\tau\beta\int_{N^\delta}^\infty |H(\alpha)|\,\mathrm{d}_\tau\alpha.$$

By Schwarz's inequality and (3.18), the β-integral here is $O(X\log X)$. For the α-integral, note that whenever $\lambda>0$, then

$$\lambda\int_y^{y+1/\lambda}|h(\lambda\alpha)|^2\,\mathrm{d}\alpha=\int_0^1|h(\alpha)|^2\,\mathrm{d}\alpha\ll X\log X,$$

irrespective of $y\in\mathbb{R}$. Hence, we may split the range $[N^\delta,\infty)$ into intervals of length $1/\lambda$ to deduce via (2.2) and (2.4) that

$$\int_{N^\delta}^\infty |h(\lambda\alpha)|^2\,\mathrm{d}_\tau\alpha\ll\tau^{-1}N^{-\delta}X\log X.$$

We apply this bound with $\lambda=\lambda_1$ and $\lambda=\lambda_2$. Recalling that $X\asymp N$ in the current context, another application of Schwarz's inequality reveals that

$$J(\mathfrak{T})\ll MN^{2-\delta}(\log N)^2, \tag{4.5}$$

which is more than is required.

For the set $\mathfrak{U}$, we argue similarly, but this time one estimates $\Phi(\alpha-\beta)$ using Lemma 4.2. The remaining double integral can be taken over $\mathbb{R}^2$, and is then readily reduced to four integrals of the type (3.18). This yields

$$J(\mathfrak{U}) \ll MN^2(\log N)^2 \exp(-2\kappa(\log M)^{3-2\gamma}). \tag{4.6}$$

Rather more care is required for $\mathfrak{V}$. The treatment begins with (4.4) and the substitution $\beta=\alpha+\zeta$. With a trivial bound for $\Phi(\zeta)$ and (2.4), we infer that

$$J(\mathfrak{V}) \ll M \int_{\mathfrak{c}} |H(\alpha)| \int_{-N^{\delta-1}}^{N^{\delta-1}} |H(\alpha+\zeta)| w_\tau(\alpha+\zeta)\, \mathrm{d}\zeta\, \mathrm{d}_\tau\alpha.$$

Reverse the order of integrations, and then apply Schwarz's inequality to the integral over $\mathfrak{c}$ to bring in the integral (3.19). This implicitly applies the interference estimates from the work of §3.4, and reveals that

$$J(\mathfrak{V}) \ll MX^{4/3+\varepsilon} \int_{-N^{\delta-1}}^{N^{\delta-1}} \Big(\int |H(\alpha+\zeta)w_\tau(\alpha+\zeta)|^2\, \mathrm{d}_\tau\alpha\Big)^{1/2} \mathrm{d}\zeta.$$

A trivial estimate for $h(\lambda_2(\alpha+\zeta))$, in combination with (2.2) and (3.18), bounds the inner integral by

$$\tau^2X^2 \int_{-\infty}^{\infty} |h(\lambda_1(\alpha+\zeta))|^2 w_\tau(\alpha+\zeta)\, \mathrm{d}\alpha \ll \tau^2X^3 \log X,$$

which then implies the bound

$$J(\mathfrak{V}) \ll \tau MN^{11/6+\delta+\varepsilon}. \tag{4.7}$$

The conclusion of Lemma 4.1 follows from (4.4), (4.5), (4.6) and (4.7).

4.3. *Potential applications*

The method exposed here is particularly useful if the test sequence ν_m is uniformly distributed modulo 1, as is the case with the example (4.1). The Fourier series $\Phi(\zeta)$ then peaks only at $\zeta=0$. The mean square method produces a double integral that Φ collapses to an expression reminiscent of a one-dimensional situation, but with two sets of generating functions. This far, there is a strong resemblence to the analysis via Plancherel's identity. The success of the method then depends on the savings that one can obtain for $\Phi(\zeta)$, and in the case (4.1) this is our Lemma 4.2.

At least in principle, the method is also applicable when ν_m runs through an arithmetic sequence, such as the values of a polynomial, but then $\Phi(\zeta)$ may have large values when ζ is in some set of major arcs $\mathfrak{M}(Q)$. Yet, at the cost of extra complication in detail, the method can still be pressed home when the initial estimations stemming from a suitable analogue of Lemma 4.2 turn out to be successful. Perelli [32] is an example where these ideas were used, and one could obtain a weaker version of Theorem 1.6 along these lines as well. However, for polynomial sequences in particular, the methods developed in I - VI are more promising, and we now turn to their initiation in the context of diophantine inequalities.

5. Fourier analysis of exceptional sets

5.1. *An illustrative example*

The work of chapter 4 depends on an analysis of the Fourier series of the test sequence. From now on we take a different point of view, and examine the Fourier series of the *exceptional set* to a representation problem.

Most of the results stated in chapter 1 that we have not yet established have a common flavour. One investigates a diophantine inequality involving a (large) parameter ν, and it is expected that there are many solutions. This expectation is tested on average over the values ν_m of a positive polynomial (Theorems 1.2, 1.3, 1.6, 1.8). There may be exceptions to the anticipated behaviour, but these are characterised by an analytic inequality: the integral over the complementary compositum must be unexpectedly large. This gives a precise meaning to an "exceptional value" of m. These numbers form a set $\mathcal{Z}$, and we consider the exponential sum

$$\sum_{m\in\mathcal{Z}} e(\alpha\nu_m).$$

This is no longer a classical Weyl sum, as was the case with (4.3), but whenever the sum reappears within moment estimates, one can restore the polynomial structure through an enveloping argument. This idea has been explored in I, II, III and V, for diophantine equations, but there is little difficulty to adapt the principal ideas to the wider context. The introductory section of I contains a detailed account of the method to which we have nothing to add. Instead, we introduce the reader to the basic strategy by working through an illustrative example that can be handled from scratch. This will ultimately yield a proof of Theorem 1.2.

Thus, we are now concerned with the weighted counter $\rho_6^*(\tau,\nu)$, we suppose that $1\le\nu\le N$, and that $X\asymp N^{1/3}$ is chosen in accordance with (2.11). We can apply Lemma 2.4, the infrastructure of which also fixes the central interval and the complementary compositum $\mathfrak{c}$. The result is that

$$\rho_6^*(\tau,\nu)=\frac{\Gamma(\frac{4}{3})^6\tau\nu}{(\lambda_1\ldots\lambda_6)^{1/3}}+\int_{\mathfrak{c}}F(\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha+O(1+\tau N^{8/9}). \tag{5.1}$$

where $F(\alpha) = f(\lambda_1\alpha)\dots f(\lambda_6\alpha)$. We will attempt to mimic the amplification procedures in §3.3, and also the interference estimate. It is important to observe that the latter can be performed without averaging, and we therefore begin with this part.

Let $\mathfrak{m}, \mathfrak{M}$ be defined as in §3.2. Following the pattern of §3.3, we write

$$\mathfrak{D} = \{\alpha \in \mathfrak{c} : \ \lambda_j\alpha \in \mathfrak{M}\ (1 \le j \le 6)\}$$

for the amplifying set, and slice it into the subsets $\mathfrak{D}(Q_1, Q_2)$ introduced in (3.14). Let

$$\widetilde{T}(N) = \min(T_{\lambda_1/\lambda_2}(\varepsilon_0 N X^{-3/2}),\ (\log\log N)^4).$$

By (3.13) and Theorem 2.8, the bound

$$\int_{\mathfrak{D}(Q_1,Q_2)} |f(\lambda_1\alpha)f(\lambda_2\alpha)|^4 \,\mathrm{d}_\tau\alpha \ll X^8 N^{-1}(Q_1Q_2)^{-1/3}\widetilde{T}(N)^{-1}$$

holds throughout the relevant range $1 \le Q_1, Q_2 \le X^{3/4}$. We may therefore sum over dyadic ranges for Q_1 and Q_2 to confirm that

$$\int_{\mathfrak{D}} |f(\lambda_1\alpha)f(\lambda_2\alpha)|^4 \,\mathrm{d}_\tau\alpha \ll N^{5/3}\widetilde{T}(N)^{-1}. \tag{5.2}$$

Also, by (3.9) one has

$$\int_{\mathfrak{D}} |f(\lambda_j\alpha)|^{16/3} \,\mathrm{d}_\tau\alpha \ll X^{7/3} \quad (3 \le j \le 6), \tag{5.3}$$

and an application of Hölder's inequality combined with (5.2) and (5.3) yields the final estimate

$$\int_{\mathfrak{D}} |f(\lambda_1\alpha)\dots f(\lambda_6\alpha)| \,\mathrm{d}_\tau\alpha \ll N\widetilde{T}(N)^{-1/4}. \tag{5.4}$$

Another subset of $\mathfrak{c}$ needs to be removed before an averaging process can be launched. Let $I \subset \{1, 2, \dots, 6\}$ be a set with 4 elements, and let

$$\mathfrak{E}(I) = \{\alpha \in \mathfrak{c}\backslash\mathfrak{D} : \ \lambda_i\alpha \in \mathfrak{M}\ (i \in I)\}.$$

We write $\mathfrak{E}$ for the union of all $\mathfrak{E}(I)$. By symmetry, there is no loss of generality in considering the special case where $I = \{3, 4, 5, 6\}$. Then, since $\alpha \notin \mathfrak{D}$, at least one of $\lambda_1\alpha$ and $\lambda_2\alpha$ lies in $\mathfrak{m}$, and consequently, the bound $f(\lambda_1\alpha)f(\lambda_2\alpha) \ll X^{7/4+\varepsilon}$ follows from (3.4). By Hölder's inequality and (3.9), we therefore obtain

$$\int_{\mathfrak{E}(I)} |F(\alpha)| \,\mathrm{d}_\tau\alpha \ll X^{7/4+\varepsilon} \prod_{j=3}^{6} \Big(\int_{\lambda_j\alpha\in\mathfrak{M}} |f(\lambda_j\alpha)|^4 \,\mathrm{d}_\tau\alpha \Big)^{1/4} \ll X^{11/4+\varepsilon}.$$

We conclude that

$$\int_{\mathfrak{E}} |F(\alpha)| \, \mathrm{d}_\tau \alpha \ll X^{11/4+\varepsilon}. \tag{5.5}$$

Now let $\mathfrak{e} = \mathfrak{c} \backslash (\mathfrak{D} \cup \mathfrak{E})$ and write

$$H_\tau(\nu) = \int_{\mathfrak{e}} F(\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha. \tag{5.6}$$

Inserting (5.4), (5.5) and (5.6) into the initial formula (5.1) for $\rho_6^*(\tau, \nu)$, in the restricted range $(\log N)^{-1} \le \tau \le 1$ we deduce that

$$\rho_6^*(\tau, \nu) = \frac{\Gamma(\frac{4}{3})^6 \tau\nu}{(\lambda_1 \dots \lambda_6)^{1/3}} + H_\tau(\nu) + O(N\widetilde{T}(N)^{-1/4}) \tag{5.7}$$

holds uniformly in $1 \le \nu \le N$.

We are ready to define the exceptional set. Let ϕ denote a positive integral quadratic polynomial, as in Theorem 1.2. Also, let M be the positive solution of the equation $\phi(2M) = N$. Then, for large N, the $\phi(m)$ with $M < m \le 2M$ are positive integers, and we define

$$\mathcal{Z}(M) = \{M < m \le 2M : \, |H_\tau(\phi(m))| \ge N/\log\log N\}.$$

We remark that when $M < m \le 2M$, but $m \notin \mathcal{Z}(M)$, then (5.7) yields the asymptotic formula

$$\rho_6^*(\tau, \phi(m)) = \frac{\Gamma(\frac{4}{3})^6 \tau\phi(m)}{(\lambda_1 \dots \lambda_6)^{1/3}} + O(N\widetilde{T}(N)^{-1/4}). \tag{5.8}$$

Hence it remains to establish a good bound for the number $Z = \operatorname{card} \mathcal{Z}(M)$.

5.2. *A quadratic average*

When $m \in \mathcal{Z}(M)$, define the complex number η_m by means of the equation $\eta_m H_\tau(\phi(m)) = |H_\tau(\phi(m))|$, and then write

$$K(\alpha) = \sum_{m \in \mathcal{Z}(M)} \eta_m e(-\alpha\phi(m)). \tag{5.9}$$

From the definitions of $H_\tau(\nu)$ and $\mathcal{Z}(M)$, we have

$$\frac{ZN}{\log\log N} \le \sum_{m \in \mathcal{Z}(M)} |H_\tau(\phi(m))| = \int_{\mathfrak{e}} F(\alpha) K(\alpha) \, \mathrm{d}_\tau \alpha. \tag{5.10}$$

This is the essential step: an upper bound for the size of the exceptional set is provided by an integral, and $K(\alpha)$ inherits the arithmetical structure of the test sequence.

Lemma 5.1. *Let $\lambda > 0$ be a fixed real number, and let $\mathcal{Z} \subset [M, 2M] \cap \mathbb{N}$ be a set of Z elements. Let ϕ be a positive polynomial of degree at least 2. For $0 < \tau \le 1$, let U_τ denote the number of solutions of the inequality*

$$|\phi(m_1) - \phi(m_2) + \lambda(x_1^3 - x_2^3)| < \tau, \tag{5.11}$$

with $m_j \in \mathcal{Z}$ and $1 \le x_j \le X$. Then

$$U_\tau \ll XZ + X^\varepsilon Z^2. \tag{5.12}$$

If ϕ is an integral polynomial, then one has also

$$U_\tau \ll XZ + X^{2+\varepsilon}. \tag{5.13}$$

Proof. When $m_1 = m_2$, the inequality (5.11) reduces to $|x_1^3 - x_2^3| < \tau/\lambda$, which has $O(X)$ solutions with $1 \le x_j \le X$. The number of solutions of this type to be counted is therefore $O(XZ)$. When $m_1 \ne m_2$, on the other hand, one has $|\phi(m_1) - \phi(m_2)| \gg M$. Hence, for any of the $O(X^2)$ possible choices for x_1, x_2, the inequality (5.11) may have no solution with $m_1 \ne m_2$ and $m_j \in \mathcal{Z}$, or else reduces to at most two equations

$$\phi(m_1) - \phi(m_2) = u, \quad \phi(m_1) - \phi(m_2) = u + 1, \tag{5.14}$$

for some $u \in \mathbb{Z}$ with $M \ll |u| \ll X^3$. When ϕ is an integral polynomial, a divisor function estimate shows that (5.14) leaves at most $O(X^\varepsilon)$ choices for m_1 and m_2. The number of solutions with $m_1 \ne m_2$ is thus at most $O(X^{2+\varepsilon})$ in this case, and this completes the proof of (5.13).

In order to establish (5.12), we count the solutions with $m_1 \ne m_2$ in a different way. There are $O(Z^2)$ possible choices for such m_1, m_2, and for each fixed such pair, the inequality (5.11) reduces to $|x_1^3 - x_2^3 - \kappa| < \tau/\lambda$, for a suitable number κ satisfying $\kappa \gg M$. Again, a divisor function estimate suffices to conclude that the number of solutions of this last inequality, in integers x_1, x_2 satisfying $1 \le x_j \le X$, is at most $O(X^\varepsilon)$. On assembling this bound together with our earlier estimate for the number of diagonal solutions, we confirm (5.12). □

As an immediate consequence of this lemma, we infer from (2.2), (2.4) and (5.9) that when λ is any one of the numbers λ_j, then

$$\int |K(\alpha) f(\lambda\alpha)|^2 \, \mathrm{d}_\tau \alpha \ll ZX + X^{2+\varepsilon}. \tag{5.15}$$

An estimate for Z can now be obtained through (5.10) and (5.15). Let $\mathfrak{a} \subset \mathbb{R}$ be a measurable set, and write

$$\mathcal{I}(\mathfrak{a}) = \int_{\mathfrak{a}} |f(\lambda_1 \alpha) \dots f(\lambda_6 \alpha) K(\alpha)| \, \mathrm{d}_\tau \alpha.$$

An estimate for $\mathcal{I}(\mathfrak{e})$ is needed, and this is accomplished by an individual treatment of various subsets of $\mathfrak{e}$, specifically

$$\mathfrak{d} = \{\alpha \in \mathfrak{e} :\ \lambda_j \alpha \in \mathfrak{m}\ (1 \le j \le 6)\}$$

and, when $i, j, l \in \{1, 2, \ldots, 6\}$ are distinct, also

$$\mathfrak{e}_{ij}(l) = \{\alpha \in \mathfrak{e} :\ \lambda_i \alpha \in \mathfrak{m},\ \lambda_j \alpha \in \mathfrak{m},\ \lambda_l \alpha \in \mathfrak{M}\}.$$

It is important to observe that $\mathfrak{e}$ is the union of $\mathfrak{d}$ and the $\mathfrak{e}_{ij}(l)$. To see this, consider $\alpha \in \mathfrak{e}\backslash\mathfrak{d}$. Then, there exists at least one l with $\lambda_l \alpha \in \mathfrak{M}$. Since α is neither in $\mathfrak{D}$ nor $\mathfrak{E}$, there are two distinct i, j with $\lambda_i \alpha \in \mathfrak{m}$, $\lambda_j \alpha \in \mathfrak{m}$, whence $\alpha \in \mathfrak{e}_{ij}(l)$, as required.

By Hölder's inequality,

$$\mathcal{I}(\mathfrak{d}) \ll \Big(\int |K f_1|^2 \, \mathrm{d}_\tau \alpha \Big)^{1/2} \prod_{j=2}^{5} \Big(\int_{\mathfrak{m}} |f_j|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/8} \sup_{\lambda_6 \alpha \in \mathfrak{m}} |f_6(\alpha)|.$$

Here and hereafter, we write $f_j = f(\lambda_j \alpha)$. By (3.4), (3.10) and (5.15), it follows that

$$\mathcal{I}(\mathfrak{d}) \ll X^{13/4} (ZX + X^{2+\varepsilon})^{1/2} (\log X)^{\varepsilon - 5/4}. \tag{5.16}$$

Also, by (3.4), one has $f_2 f_1 \ll X^{3/2+\varepsilon}$ on $\mathfrak{e}_{12}(3)$, so by Hölder's inequality,

$$\begin{aligned} \mathcal{I}(\mathfrak{e}_{12}(3)) \ll X^{3/2+\varepsilon} & \Big(\int |K f_6|^2 \, \mathrm{d}_\tau \alpha \Big)^{1/2} \Big(\int |f_5|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/8} \\ & \times \Big(\int |f_4|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/8} \Big(\int_{\mathfrak{M}} |f_3|^4 \, \mathrm{d}_\tau \alpha \Big)^{1/4}. \end{aligned}$$

Employing (3.10) once again, we find that the eighth moments of f_5 and f_4 are bounded by $O(X^5)$. The other factors can also be estimated via (5.15) and (3.9), and thus

$$\mathcal{I}(\mathfrak{e}_{12}(3)) \ll X^{3+\varepsilon} (ZX + X^{2+\varepsilon})^{1/2}, \tag{5.17}$$

a bound superior to (5.16). Also, by symmetry, this last bound holds for any other $\mathfrak{e}_{ij}(l)$ in place of $\mathfrak{e}_{12}(3)$. Hence, the bound (5.16) remains valid with $\mathfrak{d}$ replaced by $\mathfrak{e}$.

Estimating the right hand side of (5.10) by means of (5.16) and (5.17), we deduce that

$$\frac{ZN}{\log \log N} \ll X^{13/4} (ZX + X^{2+\varepsilon})^{1/2} (\log X)^{\varepsilon - 5/4}.$$

But $N \asymp X^3$, whence it now follows that

$$Z \ll X^{3/2} (\log X)^{\varepsilon - 5/2} \ll M (\log M)^{\varepsilon - 5/2}.$$

In particular, the asymptotic formula (5.8) holds for all but $O(M(\log M)^{\varepsilon-5/2})$ of the integers m with $M < m \le 2M$. One can now apply the transference principle (2.12), with $s = 6$ and $\delta = (\log\log N)^{-1/2}$, say, to conclude that the expected asymptotic formula

$$\rho_6(\tau, \phi(m)) = \frac{2\Gamma(\frac{4}{3})^6 \tau\phi(m)}{(\lambda_1 \dots \lambda_6)^{1/3}} + O(N\widetilde{T}(N)^{-1/8})$$

holds for all but $O(M(\log M)^{\varepsilon-5/2})$ of the integers $m \in (M, 2M]$ as well. A dyadic dissection argument completes the proof of Theorem 1.2.

5.3. *Some brief heckling*

Most of the results in I, II, III, V and VI depend on mean value estimates over exceptional sets, often in mixed form. Lemma 5.1 is only a typical example, and one may take our proof of Theorem 1.2 as a model for generalising our results on diophantine equations to the wider class of inequalities. However, it should be stressed that the estimate (5.13) applies to *integral polynomials only.* Its proof crucially depends on a divisor estimate that is otherwise not available. If the test sequence stems from a positive polynomial that is no longer integral, different methods have to be applied, and this is the main reason why in the non-integral case our exceptional set estimates are considerably weaker. We proceed by presenting two mean value estimates relating to general polynomials, and then illustrate their use within the proof of Theorem 1.6.

5.4. *An inequality involving quadratic polynomials*

The sole purpose of this section is to establish the following simple mean value theorem. Although the result is not needed until chapter 6, the method is a model for the work in §5.5 which is more relevant for our immediate needs.

Lemma 5.2. *Let $\phi \in \mathbb{R}[t]$ denote a quadratic polynomial. Suppose that $\mathcal{Z} \subset [1, M] \cap \mathbb{Z}$, and write $Z = \operatorname{card} \mathcal{Z}$. Finally, let $\Omega(M, \mathcal{Z})$ denote the number of solutions of the inequality*

$$|\phi(m_1) + \phi(m_2) - \phi(m_3) - \phi(m_4)| < 1 \tag{5.18}$$

with $m_j \in \mathcal{Z}$. Then one has $\Omega(M, \mathcal{Z}) \ll M^{1+\varepsilon} Z$.

Proof. We may write $\phi(t) = \lambda_2 t^2 + \lambda_1 t + \lambda_0$ for some real numbers $\lambda_0, \lambda_1, \lambda_2$ with $\lambda_2 \ne 0$. Given a solution of (5.18) counted by $\Omega(M, \mathcal{Z})$, let k_1 and k_2

be defined by means of the equations

$$k_j = m_1^j + m_2^j - m_3^j - m_4^j \quad (j = 1, 2). \tag{5.19}$$

The inequality (5.18) then reduces to

$$|\lambda_2 k_2 + \lambda_1 k_1| < 1. \tag{5.20}$$

Substituting from the linear equation in (5.19) into the quadratic one, we find that

$$2(m_1 - m_3 - k_1)(m_2 - m_3 - k_1) = 2m_3 k_1 + k_1^2 - k_2. \tag{5.21}$$

We note that $|k_1| \leq 2M$, and that for any given k_1, the inequality (5.20) allows only $O(1)$ possible choices for k_2. Thus, when the right hand side of (5.21) is non-zero, there are $O(MZ)$ possible choices for k_1, k_2 and m_3, and for any one of these choices, a divisor function estimate shows that there are $O(M^\varepsilon)$ possible values for m_1 and m_2 satisfying (5.21). The solutions of this type consequently contribute $O(M^{1+\varepsilon} Z)$ to $\Omega(M, \mathcal{Z})$. On the other hand, when the right hand side of (5.21) is zero, one has either $k_1 = 0$ or $m_3 = (k_2 - k_1^2)/(2k_1)$, and this implies that there are at most $O(M)$ possible choices for k_1, k_2 and m_3. For each fixed choice of this type, one finds from (5.21) that $m_j = m_3 + k_1$ for $j = 1$ or 2, whence there are at most $O(Z)$ integers m_1 and m_2 satisfying (5.21). The solutions of this second type therefore contribute at most $O(MZ)$ to $\Omega(M, \mathcal{Z})$. The conclusion of the lemma now follows. □

5.5. *An application of Vinogradov's method*

A version of Lemma 5.2 for polynomials of higher degree can be fabricated through a suitable generalisation of the idea exploited in §5.4. The problem may be addressed through an application of Vinogradov's mean value theorem. Let $J_{k,s}(M)$ denote the number of solutions of the simultaneous equations

$$\sum_{j=1}^{s} (x_j^l - y_j^l) = 0 \quad (1 \leq l \leq k),$$

with $1 \leq x_j, y_j \leq M$.

Lemma 5.3. *Let $\phi \in \mathbb{R}[t]$ denote a polynomial of degree $d \geq 3$. Let $s \geq 2$, and let $U_{\phi,s}(M)$ denote the number of solutions of the inequality*

$$\Big| \sum_{j=1}^{s} (\phi(x_j) - \phi(y_j)) \Big| < 1, \tag{5.22}$$

in integers x_j, y_j with $1 \le x_j, y_j \le M$. Then

$$U_{\phi,s}(M) \ll M^{\frac{1}{2}d(d-1)} J_{d,s}(M).$$

Proof. Motivated by the argument used to prove Lemma 5.2, we begin by writing $\phi(t) = \lambda_d t^d + \ldots + \lambda_1 t + \lambda_0$, with $\lambda_d \neq 0$. Given a solution of (5.22) counted by $U_{\phi,s}(M)$, we define $k_1, \ldots, k_d$ by

$$k_l = \sum_{j=1}^{s} (x_j^l - y_j^l). \tag{5.23}$$

Then (5.22) implies that $|\lambda_d k_d + \ldots + \lambda_1 k_1| < 1$. If $k_1, \ldots, k_{d-1}$ are determined, then this inequality leaves $O(1)$ possibilities for k_d. On noting that $|k_l| < sM^l$, we find that the number of choices for $k_1, \ldots, k_d$ is $O(M^{\frac{1}{2}d(d-1)})$, and hence

$$U_{\phi,s}(M) \ll M^{\frac{1}{2}d(d-1)} \max_{k_1,\ldots,k_d} J_{d,s,\mathbf{k}}(M),$$

where $J_{d,s,\mathbf{k}}(M)$ denotes the number of solutions of the diophantine system (5.23), with $1 \le x_j, y_j \le M$. But a well-known argument (see inequality (5.4) of Vaughan [35]) shows that $J_{d,s,\mathbf{k}}(M) \le J_{d,s}(M)$, and the lemma follows. □

An upper bound for $J_{d,s}(M)$ is now required that is of the expected order of magnitude. According to Theorem 3 of Wooley [42], there exists a constant C with the property that whenever

$$s > d^2(\log d + 2\log\log d + C), \tag{5.24}$$

one has $J_{d,s}(M) \ll M^{2s-\frac{1}{2}d(d+1)}$. Subject to the condition (5.24), one deduces from Lemma 5.3 the bound

$$U_{\phi,s}(M) \ll M^{2s-d}. \tag{5.25}$$

5.6. *Linear forms in primes, yet again*

In this section, we briefly indicate how (5.25) may be applied to establish Theorem 1.6. Thus, we now work under the hypotheses of that theorem. In particular, we suppose that λ_1/λ_2 is an algebraic irrational, and $\phi \in \mathbb{R}[t]$ is a polynomial of degree d. The argument, at the beginning, is largely similar to that proceeding in §4.1, but we need to replace the test sequence by $\nu_m = \phi(m)$, and the parameter M by $\phi(2M) = N$. In all other respects, we use the notation and work from §4.1 that is partly inherited from Lemma 2.5 and §3.4. This defines a complementary compositum, a parameter X

with $X \asymp N$, and $E_\tau(\nu)$ via (3.15). The substitute for Lemma 4.1 is the following estimate.

Lemma 5.4. *Let s denote a natural number, and suppose that* (5.25) *holds. Then, whenever $0 < \tau \le 1$ is fixed, one has*

$$\sum_{M<m\le 2M} |E_\tau(\phi(m))| \ll MN^{1-1/(6s)+\varepsilon}.$$

This implies Theorem 1.6, as we now demonstrate. By Lemma 5.4, the inequality $|E_\tau(\phi(m))| > N(\log N)^{-A}$ can hold for no more than $O(MN^{-1/(6s)+2\varepsilon})$ of the integers m with $M < m \le 2M$, and for the remaining values in this range, the relation (4.2) yields the asymptotic formula

$$\sigma^*(\tau, \phi(m)) = \frac{\tau\phi(m)}{\lambda_1\lambda_2} + O(N(\log N)^{-A}).$$

The now familiar transference principle takes this to an asymptotic formula for $\sigma(\tau, \phi(m))$, outside an exceptional set that still has no more than $O(MN^{-1/(6s)+2\varepsilon})$ members. But $M \asymp N^{1/d}$, and (5.25) holds whenever s satisfies (5.24). Consequently, a dyadic dissection argument delivers the conclusion of Theorem 1.6 with a permissible value of δ satisfying $\delta = \frac{1}{6} + O(\log\log d/\log d)$.

It remains to prove Lemma 5.4. Define the numbers η_m by putting $\eta_m = 0$ when $E_\tau(\phi(m)) = 0$, and otherwise via the equation $|E_\tau(\phi(m))| = \eta_m E_\tau(\phi(m))$. Also, write

$$K(\alpha) = \sum_{M<m\le 2M} \eta_m e(-\alpha\phi(m)).$$

Then, by (3.15) and Hölder's inequality,

$$\begin{aligned}\sum_{M<m\le 2M} |E_\tau(\phi(m))| &= \int_{\mathfrak{c}} K(\alpha)h(\lambda_1\alpha)h(\lambda_2\alpha)\,\mathrm{d}_\tau\alpha \\ &\le \Big(\int |K(\alpha)|^{2s}\,\mathrm{d}_\tau\alpha\Big)^{1/(2s)} J^{1/(2s)}(J_1J_2)^{\frac{1}{2}(1-1/s)},\end{aligned}$$

where

$$J = \int_{\mathfrak{c}} |h(\lambda_1\alpha)h(\lambda_2\alpha)|^2\,\mathrm{d}_\tau\alpha, \qquad J_l = \int |h(\lambda_l\alpha)|^2\,\mathrm{d}_\tau\alpha.$$

A consideration of the underlying diophantine inequality reveals that

$$\int |K(\alpha)|^{2s}\,\mathrm{d}_\tau\alpha \le U_{\phi,s}(M),$$

and we may now apply (5.25), (3.19) and (3.18) to infer that

$$\sum_{M<m\le 2M} |E_\tau(\phi(m))| \ll (M^{2s-d})^{1/(2s)}(N^{8/3+\varepsilon})^{1/(2s)}(N\log N)^{1-1/s}.$$

The proof of Lemma 5.4 is completed by recalling that $M \asymp N^{1/d}$.

Note that when ϕ is an integral polynomial, the work of Ford [20] gives much better bounds for $U_{\phi,s}(M)$, and there is then no essential difficulty in improving the exceptional set estimate to a full analogue of Theorem 1 of II. On the other hand, it seems more difficult to relax the hypothesis that λ_1/λ_2 be algebraic. If λ_1/λ_2 is merely supposed to be irrational, one might have to accept weaker exceptional set estimates.

6. Outstanding arts

6.1. *Smooth cubic Weyl sums*

This chapter is devoted to all the remaining results concerned with additive representation in thin sequences. In the next section we consider diagonal cubic forms in six variables, and we establish Theorem 1.3 by applying a complementary compositum estimate, the proof of which is the central focus of §6.3. We then pause to discuss diagonal cubics in five variables, the topic of Theorem 1.4, and a situation in which we are restricted to contemplating only lower bound estimates via the methods of IV. The argument here makes use of an asymptotic lower bound for the number of solutions of the diophantine inequality in question, a matter we defer to §6.5, incorporated into a mean value involving the exceptional set within §6.4. Some preparatory work concerning smooth Weyl sums of higher degree leads from §6.6 to the proof of Theorem 1.8 in §6.7. Finally, in §6.8, we consider prime numbers close to diagonal forms using the methods of VI, completing our journey with the proof of Theorem 1.9. All of these discussions are dependent, in some way or other, on mean value estimates for smooth Weyl sums. We finish this section by recording here those estimates that are needed to establish Theorems 1.3 and 1.4.

Define

$$\mathcal{A}(P,R) = \{x\in\mathbb{N} : x\le P,\ p|x \Rightarrow p\le R\}.$$

Also, let $\eta>0$ denote a small real number, and write

$$g(\alpha) = \sum_{x\in\mathcal{A}(X,X^\eta)} e(\alpha x^3). \tag{6.1}$$

Lemma 6.1. *Let u be one of 6 and 77/10. Then, there exists a real number $\eta_0 > 0$ such that, whenever $0 < \eta \le \eta_0$, one has the estimate*

$$\int_0^1 |g(\alpha)|^u \, \mathrm{d}\alpha \ll P^{\mu_u},$$

where $\mu_6 = 3.2495$ and $\mu_{77/10} = 4.7$.

The permissible exponent μ_u claimed in Lemma 6.1 is a consequence of Theorem 1.2 of Wooley [43] when $u = 6$, and is a special case of Theorem 2 of Brüdern and Wooley [16] when $u = 77/10$. In fact, in the case $u = 6$, we have rounded up; for a microscopically better bound see [43].

6.2. *Senary cubic forms*

In this and the next section we establish Theorem 1.3. Let $\lambda_1, \dots, \lambda_6$ be positive real numbers with λ_1/λ_2 irrational. As always, our leading parameter is N, and X is chosen in accordance with (2.11), so that $X^3 \asymp N$. With $f(\alpha)$ and $g(\alpha)$ defined by (2.10) and (6.1), and fixed $0 < \tau \le 1$, we consider the integral

$$\rho^*(\tau, \nu) = \int f(\lambda_1\alpha) f(\lambda_2\alpha) g(\lambda_3\alpha) \dots g(\lambda_6\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha. \tag{6.2}$$

By considering the underlying diophantine inequality, it follows that for all $\nu \in \mathbb{R}$ one has

$$\rho_6(\tau, \nu) \ge \rho^*(\tau, \nu). \tag{6.3}$$

The central interval is chosen as

$$\mathfrak{C} = \{\alpha \in \mathbb{R} : |\alpha| \le (\log N)^{1/4} N^{-1}\}. \tag{6.4}$$

Then, as a consequence of Lemma 8.5 of Wooley [39], there exists a number $C = C(\eta) > 0$ such that, whenever $\alpha \in \mathfrak{C}$, one has

$$f(\lambda_1\alpha) f(\lambda_2\alpha) g(\lambda_3\alpha) \dots g(\lambda_6\alpha) - C v(\lambda_1\alpha) \dots v(\lambda_6\alpha) \ll X^6 (\log X)^{-1/2},$$

where v is defined in (2.17). Now, much as in the proof of Lemma 2.4, it follows that

$$\begin{aligned} &\int_{\mathfrak{C}} f(\lambda_1\alpha) f(\lambda_2\alpha) g(\lambda_3\alpha) \dots g(\lambda_6\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha \\ &\quad = C \int_{-\infty}^{\infty} v(\lambda_1\alpha) \dots v(\lambda_6\alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha + O(X^3 (\log X)^{-1/4}). \end{aligned}$$

The integral on the right hand side here has also been evaluated in the course of the proof of Lemma 2.4, the result being $\Gamma(\frac{4}{3})^6 (\lambda_1 \dots \lambda_6)^{-1/3} \tau\nu$

for $0 \le \nu \le N$. In particular, we may conclude from the above that whenever $\frac{1}{10}N \le \nu \le N$, then one has

$$\int_{\mathfrak{C}} f(\lambda_1\alpha)f(\lambda_2\alpha)g(\lambda_3\alpha)\dots g(\lambda_6\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha \ge 2cN, \tag{6.5}$$

where $c > 0$ denotes a certain positive constant independent of ν and N.

For the treatment of the complementary compositum $\mathfrak{c} = \mathbb{R}\backslash\mathfrak{C}$, we engineer an amplification procedure that is quite similar to the one used in §3.3. The amplifier here will be

$$\mathfrak{D} = \{\alpha \in \mathfrak{c} : \lambda_1\alpha \in \mathfrak{M}(X^{3/4}),\ \lambda_2\alpha \in \mathfrak{M}(X^{3/4})\}.$$

We show in the next section that

$$\int_{\mathfrak{D}} |f(\lambda_1\alpha)f(\lambda_2\alpha)g(\lambda_3\alpha)\dots g(\lambda_6\alpha)|\,\mathrm{d}_\tau\alpha = o(N), \tag{6.6}$$

a conclusion that for the remainder of this section we take as granted.

On the complement $\mathfrak{d} = \mathfrak{c}\backslash\mathfrak{D}$, the averaging method described in §5.2 is required. Let ϕ be a positive quadratic polynomial. For large N, let M be the unique positive solution of $\phi(2M) = N$. Then, for any m with $M < m \le 2M$, we have $\frac{1}{10}N \le \phi(m) \le N$. We define $\mathcal{Z}(M)$ to be the set of integers m with $M < m \le 2M$ for which $\rho_6(\tau, \phi(m)) < cN$, where c is the number introduced in (6.5). In addition, we write $Z(M) = \operatorname{card}\mathcal{Z}(M)$. We note that Theorem 1.3 follows from a dyadic dissection argument, once one has established the bound

$$Z(M) \ll M^{23/27}. \tag{6.7}$$

Write

$$H(\nu) = \int_{\mathfrak{d}} f(\lambda_1\alpha)f(\lambda_2\alpha)g(\lambda_3\alpha)\dots g(\lambda_6\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha.$$

Then, one finds from (6.2), (6.3), (6.5) and (6.6) that for $m \in \mathcal{Z}(M)$, one has $|H(\phi(m))| \gg N$. For $m \in \mathcal{Z}(M)$, define η_m through the equation $|H(\phi(m))| = \eta_m H(\phi(m))$, and then define $K(\alpha)$ by means of (5.9). By the argument that delivered (5.10) we now infer that

$$NZ(M) \ll \int_{\mathfrak{d}} |f(\lambda_1\alpha)f(\lambda_2\alpha)g(\lambda_3\alpha)\dots g(\lambda_6\alpha)K(\alpha)|\,\mathrm{d}_\tau\alpha. \tag{6.8}$$

In the next section we show that

$$\int_{\mathfrak{d}} |f(\lambda_1\alpha)f(\lambda_2\alpha)g(\lambda_3\alpha)\dots g(\lambda_6\alpha)|^{4/3}\,\mathrm{d}_\tau\alpha \ll X^{43/9-2v}, \tag{6.9}$$

where $v = 10^{-6}$. Equipped with this estimate, we may apply Hölder's inequality to the right hand side of (6.8) and bound the fourth moment of $K(\alpha)$ by utilising Lemma 5.2. This yields

$$NZ(M) \ll (M^{1+\varepsilon}Z(M))^{1/4}X^{43/12-v},$$

and the desired conclusion (6.7) follows.

6.3. *Two technical estimates*

For notational convenience, we now write $f_j = f(\lambda_j\alpha)$ and $g_j = g(\lambda_j\alpha)$, and we define

$$J(\mathfrak{a}) = \int_{\mathfrak{a}} |f_1 f_2 g_3 g_4 g_5 g_6|^{4/3}\,\mathrm{d}_\tau\alpha.$$

We split $\mathfrak{d}$ into the three subsets

$$\mathfrak{d}_1 = \{\alpha \in \mathfrak{c} : \lambda_1\alpha \in \mathfrak{m},\ \lambda_2\alpha \in \mathfrak{M}\},$$
$$\mathfrak{d}_2 = \{\alpha \in \mathfrak{c} : \lambda_2\alpha \in \mathfrak{m},\ \lambda_1\alpha \in \mathfrak{M}\},$$
$$\mathfrak{d}_3 = \{\alpha \in \mathfrak{c} : \lambda_1\alpha \in \mathfrak{m},\ \lambda_2\alpha \in \mathfrak{m}\},$$

where, as on earlier occasions, we put $\mathfrak{M} = \mathfrak{M}(X^{3/4})$, $\mathfrak{m} = \mathfrak{m}(X^{3/4})$. By making use of (3.4) in order to estimate f_1, an application of Hölder's inequality shows that

$$J(\mathfrak{d}_1) \ll X^{1+\varepsilon}\Big(\int_{\lambda_2\alpha\in\mathfrak{M}} |f_2|^4\,\mathrm{d}_\tau\alpha\Big)^{1/3}\prod_{j=3}^{6}\Big(\int |g_j|^8\,\mathrm{d}_\tau\alpha\Big)^{1/6}.$$

The eighth moment of g_j is $O(X^5)$; this can be seen either via (3.10), on considering the underlying diophantine inequality, or by reference to Lemma 6.1. By (3.9), the restricted fourth moment of f_2 is $O(X^{1+\varepsilon})$, and thus we deduce that $J(\mathfrak{d}_1) \ll X^{14/3+\varepsilon}$. By symmetry, the same bound holds also for $J(\mathfrak{d}_2)$. In order to estimate $J(\mathfrak{d}_3)$, meanwhile, we again use (3.4) to bound f_1 and $|f_2|^{8/9}$, and then apply Hölder's inequality, thereby confirming that

$$J(\mathfrak{d}_3) \ll X^{5/3+\varepsilon}\Big(\int |f_2|^4\,\mathrm{d}_\tau\alpha\Big)^{1/9}\prod_{j=3}^{6}\Big(\int |g_j|^6\,\mathrm{d}_\tau\alpha\Big)^{2/9}.$$

Here the fourth moment of $|f_2|$ is $O(X^{2+\varepsilon})$, as one finds by considering the underlying diophantine inequality, or by reference to Hua's lemma. The sixth moments of g_j are each $O(X^{\mu_6})$, by Lemma 6.1. This then yields the estimate $J(\mathfrak{d}_3) \ll X^{43/9-2v}$. In combination with our earlier bounds for $J(\mathfrak{d}_1)$ and $J(\mathfrak{d}_2)$, the estimate (6.9) is confirmed on noting that $J(\mathfrak{d}) = J(\mathfrak{d}_1) + J(\mathfrak{d}_2) + J(\mathfrak{d}_3)$.

We now turn to the proof of (6.6). Let $T = T_{\lambda_1/\lambda_2}((\log N)^{1/4})$ be defined via (2.22). Then $T \to \infty$ as $N \to \infty$. Now let

$$\mathfrak{E} = \{\alpha \in \mathfrak{D} : \lambda_j\alpha \in \mathfrak{M}(T^{1/4}) \text{ for } j = 1, 2\}, \quad \mathfrak{e} = \mathfrak{D}\backslash\mathfrak{E}.$$

Then, one finds from Theorem 2.8 that

$$\int_{\mathfrak{E}} |f_1f_2g_3g_4g_5g_6| \,\mathrm{d}_\tau\alpha \ll X^6 \int_{\mathfrak{E}} \mathrm{d}_\tau\alpha \ll X^3T^{-1/2}, \tag{6.10}$$

which is acceptable. Moreover, on combining (3.4) with the major arc upper bound for $|f_1f_2|$ derived from (3.13), we infer that

$$\sup_{\mathfrak{e}} |f_1f_2| \ll X^2T^{-1/12}.$$

Now, by Hölder's inequality,

$$\begin{aligned}\int_{\mathfrak{e}} &|f_1f_2g_3g_4g_5g_6| \,\mathrm{d}_\tau\alpha \\ &\ll (\sup_{\mathfrak{e}} |f_1f_2|)^{1/44}\Big(\int_{\lambda_1\alpha\in\mathfrak{M}} |f_1|^{301/74} \,\mathrm{d}_\tau\alpha\Big)^{37/154} \\ &\quad\times \Big(\int_{\lambda_2\alpha\in\mathfrak{M}} |f_2|^{301/74} \,\mathrm{d}_\tau\alpha\Big)^{37/154} \prod_{j=3}^{6} \Big(\int |g_j|^{77/10} \,\mathrm{d}_\tau\alpha\Big)^{10/77}.\end{aligned}$$

The moments of g_j can be bounded using Lemma 6.1, and the moments of f_1, f_2 by (3.9) (where it is important to note that $\frac{301}{74} > 4$). It follows that the integral in question is $O(X^3T^{-1/528})$. The desired estimate (6.6) now follows by combining this bound with (6.10).

6.4. *The lower bound variant*

We now embark on the proof of Theorem 1.4. The method derives from IV to which we refer for a discussion of the main idea. Fundamental to its success is a lower bound for the number of solutions of a related equation or inequality in which the test sequence occurs as an additional variable. Thus, we need the following result.

Lemma 6.2. *Let $\phi \in \mathbb{R}[t]$ denote a positive quadratic polynomial, and let $\lambda_1, \dots, \lambda_5$ denote positive real numbers with $\lambda_1/\lambda_2 \notin \mathbb{Q}$. Let X be sufficiently large, and let M be a positive real number with $M \asymp X^{3/2}$ satisfying the condition that $\phi(2M) < 2\lambda_jX^3$ for all j. Finally, for any fixed $0 < \tau \leq 1$, let*

$$V = \sum \widehat{w}_\tau(\lambda_1x_1^3 + \ldots + \lambda_5x_5^3 - \phi(m)),$$

with the sum extended over $m, x_1, \ldots, x_5$ *in the ranges*

$$x_1, x_2 \le X, \quad x_3, x_4, x_5 \in \mathcal{A}(X, X^\eta), \quad M < m \le 2M.$$

Then, one has $V \gg X^2 M$.

Note that V counts solutions of the diophantine inequality

$$|\lambda_1 x_1^3 + \ldots + \lambda_5 x_5^3 - \phi(m)| < \tau, \tag{6.11}$$

with a certain non-negative weight attached. A related result occurs as Theorem 2 of Brüdern [4], but it does not cover Lemma 6.2. The cited work predates the innovations of Bentkus and Götze, and of Freeman, and therefore, one would find a lower bound for V only for a certain sequence of values of the parameter X. Secondly, in [4] the polynomial ϕ has to be a monomial. It is relatively straightforward to attend to these two problems. More importantly, however, the method in [4] is laid out only for an algebraic irrational coefficient ratio. Therefore, we spell out a proof of Lemma 6.2 in the next section.

It is a delightful exercise to deduce Theorem 1.4. Let N be a large real number, let X be defined in accordance with (2.11), and choose M as in the statement of Lemma 6.2. Next, let $\mathcal{Z}(M)$ denote the set of all m with $M < m \le 2M$ for which the diophantine inequality (6.11) has at least one solution in positive integers x_j, and put $Z = \operatorname{card} \mathcal{Z}(M)$. We write

$$\Phi(\alpha) = \sum_{M<m\le 2M} e(\alpha\phi(m)), \quad k(\alpha) = \sum_{m\in\mathcal{Z}(M)} e(\alpha\phi(m)),$$
$$G(\alpha) = f_1(\alpha) f_2(\alpha) g_3(\alpha) g_4(\alpha) g_5(\alpha).$$

Then, on considering the underlying diophantine inequality, one verifies that

$$\int G(\alpha)\Phi(-\alpha)\,\mathrm{d}_\tau\alpha = \int G(\alpha)k(-\alpha)\,\mathrm{d}_\tau\alpha. \tag{6.12}$$

By orthogonality, the left hand side here is equal to the quantity V defined in Lemma 6.2. Hence, by that lemma, it follows that the integrals in (6.12) are asymptotically bounded from below by $X^2 M$.

We now estimate the integral on the right hand side from above. With this end in view, we cover the real line by the three sets

$$\begin{aligned} \mathfrak{e}_j &= \{\alpha \in \mathbb{R} : \lambda_j\alpha \in \mathfrak{m}\} \quad (j = 1, 2), \\ \mathfrak{E} &= \{\alpha \in \mathbb{R} : \lambda_1\alpha \in \mathfrak{M},\ \lambda_2\alpha \in \mathfrak{M}\}. \end{aligned}$$

When $\mathfrak{a}$ is any one of these sets, we write

$$I(\mathfrak{a}) = \int_{\mathfrak{a}} |G(\alpha)k(\alpha)| \, \mathrm{d}_\tau \alpha.$$

By (6.12) and the discussion thereafter, it follows that

$$X^2 M \ll I(\mathfrak{e}_1) + I(\mathfrak{e}_2) + I(\mathfrak{E}). \tag{6.13}$$

It remains to establish upper bounds for $I(\mathfrak{e}_1)$, $I(\mathfrak{e}_2)$ and $I(\mathfrak{E})$. On the set $\mathfrak{E}$, we use the trivial bound $|k(\alpha)| \le Z$, and apply Hölder's inequality to infer that

$$I(\mathfrak{E}) \le Z\Big(\int_{\lambda_1 \alpha \in \mathfrak{M}} |f_1|^4 \, \mathrm{d}_\tau \alpha \Big)^{1/4} \Big(\int_{\lambda_2 \alpha \in \mathfrak{M}} |f_2|^4 \, \mathrm{d}_\tau \alpha \Big)^{1/4} \times \prod_{j=3}^{5} \Big(\int |g_j|^6 \, \mathrm{d}_\tau \alpha \Big)^{1/6}.$$

One may estimate the first two integrals by applying (3.9). An upper bound for the sixth moments of g_j is given in Lemma 6.1, and we deduce that

$$I(\mathfrak{E}) \ll X^{17/8} Z. \tag{6.14}$$

A similar argument may be used to estimate $I(\mathfrak{e}_1)$. By (3.4), we find that $f_1(\alpha) \ll X^{3/4+\varepsilon}$ whenever $\alpha \in \mathfrak{e}_1$. Hölder's inequality now reveals that

$$I(\mathfrak{e}_1) \ll X^{3/4+\varepsilon} \Big(\int |kf_2|^2 \, \mathrm{d}_\tau \alpha \Big)^{1/2} \prod_{j=3}^{5} \Big(\int |g_j|^6 \, \mathrm{d}_\tau \alpha \Big)^{1/6}.$$

As before, the sixth moments of g_j may be estimated through Lemma 6.1. Moreover, by Lemma 5.1, we have

$$\int |k(\alpha) f_2(\alpha)|^2 \, \mathrm{d}_\tau \alpha \ll X^\varepsilon Z^2 + XZ.$$

Therefore, it follows that $I(\mathfrak{e}_1) \ll X^{19/8}(Z^2 + XZ)^{1/2}$, and by symmetry, the same bound holds for $I(\mathfrak{e}_2)$. On combining these estimates with (6.13) and (6.14), we conclude that

$$X^2 M \ll X^{17/8} Z + X^{19/8} (Z^2 + XZ)^{1/2}.$$

This implies the lower bound $Z \gg M^{3/4}$, as required to complete the proof of Theorem 1.4.

6.5. *An auxiliary inequality*

In the notation of the previous section, our object is to evaluate the integral

$$V = \int G(\alpha)\Phi(-\alpha)\,\mathrm{d}_\tau\alpha. \tag{6.15}$$

Define $\mathfrak{C}$ as in (6.4). Then, just as in the discussion following the latter definition, there is little difficulty in adapting the arguments applied to prove Lemma 2.4 so as to establish here the lower bound

$$\int_{\mathfrak{C}} G(\alpha)\Phi(-\alpha)\,\mathrm{d}_\tau\alpha \gg X^2M. \tag{6.16}$$

We feel entitled by now to omit the details.

The treatment of the complementary compositum depends on a technique sometimes referred to as pruning, and made available for diophantine inequalities by Brüdern [4].

Lemma 6.3. *Let $\phi(t) = \lambda t^2 + \mu t + \zeta \in \mathbb{R}[t]$, with $\lambda \neq 0$. In addition, let $\mathfrak{K} = \{\alpha \in \mathbb{R} : \lambda\alpha \in \mathfrak{M}(X)\}$, and let $\mathfrak{k} = \mathbb{R}\backslash\mathfrak{K}$. Then given a fixed non-zero real number ω, for any $0 < \tau \leq 1$, one has*

$$\int_{\mathfrak{K}} |\Phi(\alpha)g(\omega\alpha)|^2\,\mathrm{d}_\tau\alpha \ll X^{2+\varepsilon},$$
$$\int_{\mathfrak{k}} |\Phi(\alpha)|^4|g(\omega\alpha)|^2\,\mathrm{d}_\tau\alpha \ll M^2X^{1+\varepsilon}.$$

Moreover, these estimates remain valid if f is substituted for g.

Proof. Note that ϕ, ω and τ are fixed in the current context. We may substitute $\beta = \lambda\alpha$ in both integrals. This replaces ω by ω/λ, and τ by τ/λ. Hence, we may assume that $\lambda = 1$ in the proof of this lemma. This is mostly for notational convenience. Note that we now have $\mathfrak{K} = \mathfrak{M}(X)$ and $\mathfrak{k} = \mathfrak{m}(X)$.

Next, we define the function Υ on $\mathbb{R}$ by taking

$$\Upsilon(\alpha) = (q + N|q\alpha - a|)^{-1}, \tag{6.17}$$

when $\alpha \in \mathfrak{M}(\frac{1}{5}M)$, and a and q are the unique coprime integers with $1 \leq q \leq \frac{1}{5}M$ and $|q\alpha - a| \leq M/(5N)$. We put $\Upsilon(\alpha) = 0$ for $\alpha \notin \mathfrak{M}(\frac{1}{5}M)$. Then, by Weyl's inequality and a familiar transference principle (see [35], Lemma 2.4 and Exercise 2 of §2.8), for all $\alpha \in \mathbb{R}$, one has

$$|\Phi(\alpha)|^2 \ll M^{2+\varepsilon}\Upsilon(\alpha) + M^{1+\varepsilon}. \tag{6.18}$$

Next, by Lemma 3 of Brüdern [4], one finds that whenever $1 \le Q \le \frac{1}{5}M$, one has

$$\int_{\mathfrak{M}(Q)} \Upsilon(\alpha)|g(\omega\alpha)|^2 \,\mathrm{d}_\tau\alpha \ll (QX + X^2)N^{\varepsilon-1}. \tag{6.19}$$

We choose $Q = X$ and observe that for $\alpha \in \mathfrak{M}(X)$ one has $\Upsilon(\alpha) \ge \frac{1}{2}X^{-1}$, so that $|\Phi(\alpha)|^2 \ll M^{2+\varepsilon}\Upsilon(\alpha)$. The first bound claimed in the lemma is therefore immediate. In order to establish the second bound, we begin by covering $\mathfrak{m}(X)$ by a dyadic dissection of the form $\mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)$, with $X \le Q \le \frac{1}{10}M$, and the residual set $\mathfrak{m}^* = \mathfrak{m}(X) \setminus \mathfrak{M}(\frac{1}{5}M)$. Then one finds from (6.18) that $|\Phi(\alpha)|^2 \ll M^{1+\varepsilon}$ for $\alpha \in \mathfrak{m}^*$, whence

$$\int_{\mathfrak{m}^*} |\Phi(\alpha)|^4|g(\omega\alpha)|^2 \,\mathrm{d}_\tau\alpha \ll M^{2+\varepsilon}\int |g(\omega\alpha)|^2 \,\mathrm{d}_\tau\alpha \ll XM^{2+\varepsilon}.$$

Also from (6.18), one sees that when $\alpha \in \mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)$ and $Q \le \frac{1}{10}M$, then one has $|\Phi(\alpha)|^4 \ll M^{4+\varepsilon}Q^{-1}\Upsilon(\alpha)$, and thus (6.19) yields the bound

$$\int_{\mathfrak{M}(2Q)\backslash\mathfrak{M}(Q)} |\Phi(\alpha)|^4|g(\omega\alpha)|^2 \,\mathrm{d}_\tau\alpha \ll (X + X^2Q^{-1})M^{2+\varepsilon}.$$

The second bound of the lemma now follows on adding the contribution arising from the $O(\log N)$ values of Q comprising the aforementioned dyadic dissection, and incorporating our earlier bound for the contribution stemming from $\mathfrak{m}^*$. Finally, when g is replaced by f, then the conclusion of Lemma 6.2 follows *mutatis mutandis.* □

We now consider the integral

$$\mathcal{I}(\mathfrak{a}) = \int_{\mathfrak{a}} |G(\alpha)\Phi(\alpha)| \,\mathrm{d}_\tau\alpha.$$

In view of (6.16), it suffices now to show that $\mathcal{I}(\mathfrak{c}) = o(X^2M)$, for it then follows from (6.15) that

$$V \gg X^2M, \tag{6.20}$$

as required to complete the proof of Lemma 6.2. Recall the notation of the statement of Lemma 6.3. We begin an amplification argument by considering the set

$$\mathfrak{e} = \{\alpha \in \mathfrak{c} : \lambda\alpha \in \mathfrak{m}(X)\}.$$

Then, by Hölder's inequality, one finds that

$$\mathcal{I}(\mathfrak{e}) \le \Big(\int_{\mathfrak{e}} |\Phi|^4|f_1|^2 \,\mathrm{d}_\tau\alpha\Big)^{1/4}\Big(\int |f_1|^2|f_2|^4 \,\mathrm{d}_\tau\alpha\Big)^{1/4}\prod_{j=3}^{5}\Big(\int |g_j|^6 \,\mathrm{d}_\tau\alpha\Big)^{1/6}.$$

By Schwarz's inequality, followed by applications of (3.10) and Hua's Lemma (see Vaughan [35], Lemma 2.5), we deduce that

$$\int |f_1|^2|f_2|^4 \,\mathrm{d}_\tau\alpha \le \Big(\int |f_1|^4 \,\mathrm{d}_\tau\alpha\Big)^{1/2}\Big(\int |f_2|^8 \,\mathrm{d}_\tau\alpha\Big)^{1/2} \ll X^{7/2+\varepsilon}.$$

Consequently, by Lemmata 6.1 and 6.3, it now follows that

$$\mathcal{I}(\mathfrak{e}) \ll (M^2X)^{1/4}X^{7/8+\varepsilon}(X^{\mu_6})^{1/2} = o(MX^2),$$

a bound which does not interfere with the desired conclusion (6.20).

On the remaining set, we may suppose that $\lambda\alpha \in \mathfrak{M}(X)$. We next dispose of the subsets

$$\mathfrak{E}_j = \{\alpha \in \mathfrak{c} : \lambda\alpha \in \mathfrak{M}(X) \text{ and } \lambda_j\alpha \in \mathfrak{m}(X^{3/4})\}.$$

Then, by (3.4) and Hölder's inequality, we have

$$\mathcal{I}(\mathfrak{E}_1) \ll X^{3/4+\varepsilon}\Big(\int_{\mathfrak{K}} |\Phi f_2|^2 \,\mathrm{d}_\tau\alpha\Big)^{1/2}\prod_{j=3}^{5}\Big(\int |g_j|^6 \,\mathrm{d}_\tau\alpha\Big)^{1/6}.$$

The first integral may be estimated by applying Lemma 6.3, and for the sixth moments of g_j we again make use of Lemma 6.1. We then find that

$$\mathcal{I}(\mathfrak{E}_1) \ll X^{3/4+\varepsilon}(X^{2+\varepsilon})^{1/2}(X^{\mu_6})^{1/2} \ll MX^{15/8}.$$

By symmetry, the same bound holds for $\mathcal{I}(\mathfrak{E}_2)$.

It remains to consider the amplifying set

$$\mathfrak{D} = \{\alpha \in \mathfrak{c} : \lambda\alpha \in \mathfrak{M}(X) \text{ and } \lambda_j\alpha \in \mathfrak{M}(X^{3/4}) \ (j=1,2)\}.$$

The endgame is almost identical to the one described in §6.3. If T is defined as in the discussion surrounding (6.10), then one may show that the portion of $\mathfrak{D}$, wherein $\lambda_1\alpha \in \mathfrak{M}(T^{1/4})$ and $\lambda_2\alpha \in \mathfrak{M}(T^{1/4})$ hold simultaneously, makes a contribution to $\mathcal{I}(\mathfrak{D})$ that is $O(MX^2T^{-1/2}) = o(MX^2)$. The required argument follows exactly that leading to (6.10). This leaves the two sets

$$\mathfrak{D}_j = \{\alpha \in \mathfrak{D} :\ \lambda_j\alpha \in \mathfrak{m}(T^{1/4})\} \quad (j=1,2).$$

Here we apply (3.13) to bound $f(\lambda_j\alpha)$, and thereby conclude that one has $f(\lambda_j\alpha) \ll XT^{-1/12}$ throughout $\mathfrak{D}_j$, whence

$$\sup_{\alpha\in\mathfrak{D}_1\cup\mathfrak{D}_2} |f_1f_2| \ll X^2T^{-1/12}.$$

We now infer from Hölder's inequality that

$$\mathcal{I}(\mathfrak{D}_1 \cup \mathfrak{D}_2) \le \sup_{\alpha \in \mathfrak{D}_1 \cup \mathfrak{D}_2} |f_1 f_2|^{1/8} \Big(\int_{\mathfrak{D}} |\Phi|^5 \, \mathrm{d}_\tau \alpha \Big)^{1/5} \prod_{j=3}^{5} \Big(\int |g_j|^8 \, \mathrm{d}_\tau \alpha \Big)^{1/8}$$
$$\times \Big(\int_{\mathfrak{D}} |f_1|^{70/17} \, \mathrm{d}_\tau \alpha \Big)^{17/80} \Big(\int_{\mathfrak{D}} |f_2|^{70/17} \, \mathrm{d}_\tau \alpha \Big)^{17/80}.$$

The last two integrals can be bounded with the aid of (3.9), and for the eighth moments of g_j, one may apply Lemma 6.1 combined with a trivial estimate. In addition, when $\lambda\alpha \in \mathfrak{M}(X)$, one may apply the recent work of Vaughan [36] to show that $\Phi(\alpha) \ll M\Upsilon(\lambda\alpha)^{1/2}$, with Υ defined as in (6.17). Then, a straightforward calculation reveals that

$$\int_{\mathfrak{D}} |\Phi(\alpha)|^5 \, \mathrm{d}_\tau \alpha \ll M^3.$$

Note here the important feature that there is no inflation of the estimate by an unacceptable logarithmic factor. Collecting together the estimates of this paragraph, we find that $\mathcal{I}(\mathfrak{D}_1 \cup \mathfrak{D}_2) \ll MX^2 T^{-1/96}$. In summary, we have shown that $\mathcal{I}(\mathfrak{D}) \ll MX^2 T^{-1/96}$, and hence also $\mathcal{I}(\mathfrak{c}) = o(MX^2)$. The proof of Lemma 6.2 is therefore complete.

6.6. *Additive forms of large degree*

In the last three sections of this chapter, we discuss the distribution of the values of the additive form (1.1) for larger degree. Theorem 1.8 will be established in §6.7, and Theorem 1.9 in §6.8. Here, we summarize results on smooth Weyl sums over k-th powers. The definition of $g(\alpha)$ in (6.1) is now to be replaced by the more general

$$g(\alpha) = \sum_{x \in \mathcal{A}(X, X^\eta)} e(\alpha x^k).$$

It will also be convenient to define

$$t_0(k) = \lceil \tfrac{1}{2} k (\log k + \log\log k + 1) \rceil + \lceil \tfrac{1}{2} k (1 + 1/\sqrt{\log k}) \rceil.$$

Lemma 6.4. *For any $\varepsilon > 0$ there exists $\eta_0(\varepsilon) > 0$ such that whenever $0 < \eta \le \eta_0$ and $t \in \mathbb{N}$, one has*

$$\int_0^1 |g(\alpha)|^{2t} \, \mathrm{d}\alpha \ll X^{2t-k+\Delta_t+\varepsilon},$$

where the real number Δ_t *satisfies* $\Delta_t e^{\Delta_t/k} \le k e^{1-2t/k}$. *Moreover, when* $t \ge t_0(k)$, *then*

$$\int_0^1 |g(\alpha)|^{2t} \, \mathrm{d}\alpha \ll X^{2t-k}.$$

Proof. The first estimate is the corollary to Theorem 2.1 of Wooley [41], the second is (5.2) of V. □

Lemma 6.5. *Let* $Y = X^{2/3}$. *Then, uniformly for* $\alpha \in \mathfrak{m}(Y)$, *one has* $g(\alpha) \ll X^{1-\mu}$ *for some* $\mu = \mu_k > 0$. *Moreover, for* $1 \le T \le (\log X)^2$ *and* $\alpha \in \mathfrak{m}(T)$, *one has* $g(\alpha) \ll X T^{\varepsilon - 1/(2k)}$. *Finally, for any real number* $t > 4k$, *one has*

$$\int_{\mathfrak{N}(Y)} |g(\alpha)|^t \, \mathrm{d}\alpha \ll X^{t-k}.$$

Proof. The first bound, of Weyl's type, follows from Theorem 1.4 of Wooley [40], for example. By combining Lemmata 7.2 and 8.5 of Vaughan and Wooley [37], one may confirm that

$$|g(\alpha)| \ll X(q + X^k |q\alpha - a|)^{\varepsilon - 1/(2k)}$$

whenever $q \le Y$, $|q\alpha - a| \le Y X^{-k}$ and $(a, q) = 1$. The second upper bound for $g(\alpha)$ is now transparent, and the major arc estimate follows via a straightforward calculation. □

We also present a rather general treatment for the expected main terms. Suppose that $\lambda_1, \dots, \lambda_t$ are positive real numbers. Also, define

$$X = 2(\lambda_1^{-1/k} + \dots + \lambda_t^{-1/k} + 1) N^{1/k}. \tag{6.21}$$

Then, whenever $1 \le \nu \le N$ and $0 < \tau \le 1$, and $x_j \in \mathbb{N}$ satisfy

$$|\lambda_1 x_1^k + \dots + \lambda_t x_t^k - \nu| < \tau,$$

one finds that $x_j \le X$. The central interval $\mathfrak{C}$ remains defined by (6.4). Then for any fixed $\tau \in (0, 1]$, uniformly in $1 \le \nu \le N$, it follows that whenever $t > k$ one has

$$\int_{\mathfrak{C}} g(\lambda_1 \alpha) \dots g(\lambda_t \alpha) e(-\nu\alpha) \, \mathrm{d}_\tau \alpha = c \nu^{t/k - 1} + O(X^{t-k} (\log X)^{-1/4}). \tag{6.22}$$

Here, the constant $c > 0$ is independent of ν and X. This is readily established by following the method of proof of Lemma 2.4, but the approximation of $g(\lambda_j \alpha)$ is accomplished via Lemma 8.5 of Wooley [39], for example. The reader is entitled to be spared further details.

6.7. *Proof of Theorem 1.8*

Consider the situation described in Theorem 1.8. Suppose that $1 \le \nu \le N$, and that $X \asymp N^{1/k}$ is chosen in accordance with (6.21). In particular, let $s \in \mathbb{N}$ with $s \ge \max(\frac{3}{2}t_0(k)+1,\, 4k+3)$, and consider the integral

$$\mathfrak{J}(\nu) = \int g(\lambda_1\alpha)\dots g(\lambda_s\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha, \tag{6.23}$$

where $0 < \tau \le 1$ is fixed from now on. Then by (6.22), uniformly for $\frac{1}{10}N < \nu \le N$, the central interval $\mathfrak{C}$ contributes $\gg X^sN^{-1}$ to the integral in (6.23). On the complementary compositum $\mathfrak{c}$, we first define $T = T_{\lambda_1/\lambda_2}(\log N)$ via (2.22), and then the amplifying set by

$$\mathfrak{D} = \{\alpha \in \mathfrak{c} : \lambda_1\alpha \in \mathfrak{M}(T^{1/4}),\ \lambda_2\alpha \in \mathfrak{M}(T^{1/4})\}.$$

Then, as in (6.10), one may employ Theorem 2.8 to establish that the contribution of $\mathfrak{D}$ to the integral (6.23) does not exceed $O(X^sN^{-1}T^{-1/2})$, uniformly in ν.

We are now reduced to the set $\mathfrak{d} = \mathfrak{c}\backslash\mathfrak{D}$, and here we average over the quadratic polynomial ϕ. Let M be the positive solution of $\phi(2M) = N$. Suppose that m is an integer with $M < m \le 2M$ for which

$$|\lambda_1x_1^k + \ldots + \lambda_sx_s^k - \phi(m)| < \tau$$

has no solution in natural numbers x_j. Then $\mathfrak{J}(\phi(m)) = 0$. Let $\mathcal{Z}(M)$ be the set of all such m, and write $Z(M)$ for its cardinality. Recalling what has just been said concerning the contributions of $\mathfrak{C}$ and $\mathfrak{D}$ to (6.23), it follows that for all $m \in \mathcal{Z}(M)$, one has the lower bound

$$\Big|\int_{\mathfrak{d}} g(\lambda_1\alpha)\dots g(\lambda_s\alpha)e(-\phi(m)\alpha)\,\mathrm{d}_\tau\alpha\Big| \gg X^sN^{-1}.$$

We sum over these exceptional m. Then, for suitable $\eta_m \in \mathbb{C}$ with $|\eta_m| = 1$, and with $K(\alpha)$ defined by (5.9), we infer that

$$X^sN^{-1}Z(M) \ll \int_{\mathfrak{d}} |g(\lambda_1\alpha)\dots g(\lambda_s\alpha)K(\alpha)|\,\mathrm{d}_\tau\alpha.$$

Let $Y = X^{2/3}$, and let $\mathfrak{d}_j$ be the set of all $\alpha \in \mathfrak{d}$ with $\lambda_j\alpha \in \mathfrak{m}(Y)$. Then, by Hölder's inequality and Lemma 6.5,

$$\int_{\mathfrak{d}_1} |g_1\dots g_sK|\,\mathrm{d}_\tau\alpha$$
$$\ll X^{1-\mu}\Big(\int |K|^4\,\mathrm{d}_\tau\alpha\Big)^{1/4}\prod_{j=2}^{s}\Big(\int |g_j|^{(4s-4)/3}\,\mathrm{d}_\tau\alpha\Big)^{3/(4s-4)}.$$

We estimate the first integral using Lemma 5.2. Also, since $\frac{4}{3}(s-1) \geq 2t_0(k)$, one may apply Lemma 6.4 to the moments of g_j. Thus we obtain the upper bound

$$\int_{\mathfrak{d}_1} |g_1 \dots g_s K| \, \mathrm{d}_\tau \alpha \ll (M^{1+\varepsilon} Z(M))^{1/4} X^{s-3k/4-\mu}.$$

By symmetry, the same bound holds for any other $\mathfrak{d}_j$. It therefore remains only to discuss the contribution from the set

$$\mathfrak{e} = \{\alpha \in \mathfrak{d} : \lambda_j \alpha \in \mathfrak{M}(Y) \ (1 \leq j \leq s)\}.$$

The definition of $\mathfrak{D}$ combined with Lemma 6.5 shows that for $\alpha \in \mathfrak{d}$ one has $g(\lambda_1\alpha)g(\lambda_2\alpha) \ll X^2 T^{\varepsilon-1/(8k)}$. Moreover, by applying Hölder's inequality in combination with the major arc estimate from Lemma 6.5, it is clear that

$$\int_{\mathfrak{e}} |g_3 g_4 \dots g_s| \, \mathrm{d}_\tau \alpha \ll X^{s-2-k},$$

whence

$$\int_{\mathfrak{e}} |g_1 \dots g_s K| \, \mathrm{d}_\tau \alpha \ll X^{s-k} T^{\varepsilon-1/(8k)} Z(M).$$

Assembling together the estimates of this section, we find that

$$X^s N^{-1} Z(M) \ll X^{s-k} T^{\varepsilon-1/(8k)} Z(M) + (M^{1+\varepsilon} Z(M))^{1/4} X^{s-3k/4-\mu},$$

and we may therefore conclude that $Z(M) \ll M^{1+\varepsilon} X^{-4\mu/3}$. This establishes Theorem 1.8.

6.8. *Proof of Theorem 1.9*

In this section, we discuss the diophantine inequality

$$|\lambda_1 x_1^k + \dots + \lambda_s x_s^k - p - \tfrac{1}{2}| < \tfrac{1}{2} \tag{6.24}$$

by the methods of §6.7. If p is a prime and $x_1, \dots, x_s$ are natural numbers satisfying (6.24), then

$$[\lambda_1 x_1^k + \dots + \lambda_s x_s^k] = p.$$

We choose N and X in accordance with (6.21), and set $\tau = \frac{1}{2}$. In the present context, it is appropriate to modify the definition of $h(\alpha)$ so that

$$h(\alpha) = \sum_{p \leq N} (\log p) e(p\alpha).$$

We then put

$$\mathcal{J} = \int g(\lambda_1\alpha)\dots g(\lambda_s\alpha)h(-\alpha)e(-\tfrac{1}{2}\alpha)\,\mathrm{d}_\tau\alpha. \tag{6.25}$$

This integral provides a weighted count of the solutions of (6.24) with $x_j \le X$ and $p \le N$, in which the weight is non-negative. We assume the Riemann hypothesis for Dirichlet L-functions, and proceed to show that for $s \ge \frac{8}{3}k + 2$, one has $\mathcal{J} \gg X^s$. This suffices to establish Theorem 1.9.

It will be useful to denote the contribution to (6.25) from a measurable set $\mathfrak{a} \subset \mathbb{R}$ by $\mathcal{J}(\mathfrak{a})$. In order to estimate $\mathcal{J}(\mathfrak{C})$, we choose $\nu = p + \frac{1}{2}$ and $\tau = \frac{1}{2}$ in (6.22). Multiplying by $\log p$ and summing over primes $p \le N$, we confirm the lower bound $\mathcal{J}(\mathfrak{C}) \gg X^s$. Hence, if $\mathfrak{c}$ denotes the complementary compositum, it now suffices to show that $\mathcal{J}(\mathfrak{c}) = o(X^s)$. It is only at this point that the Riemann hypothesis is required, and it is invoked through Lemma 2 of Brüdern and Perelli [14]. As a consequence of the latter, we have

$$\sup_{\alpha\in\mathfrak{m}(N^{1/6})} |h(\alpha)| \ll N^{5/6+\varepsilon}.$$

Also, when $\alpha \in \mathfrak{M}(N^{1/6})$, one has $h(\alpha) \ll \Upsilon^*(\alpha)$, where for $\alpha = a/q + \beta$ with $(a,q) = 1$, $1 \le q \le N^{1/6}$ and $|\beta| \le q^{-1}N^{-5/6}$, the function Υ^* is defined by

$$\Upsilon^*(\alpha) = N\varphi(q)^{-1}(1 + N|\beta|)^{-1}.$$

Here $\varphi(q)$ denotes Euler's totient.

We split $\mathfrak{c}$ into various subsets of which we first treat

$$\mathfrak{e} = \{\alpha \in \mathfrak{c} : \alpha \in \mathfrak{m}(N^{1/6})\}.$$

On this set, Hölder's inequality yields

$$\mathcal{J}(\mathfrak{e}) \ll N^{5/6+\varepsilon} \prod_{j=1}^{s} \Big(\int |g_j|^s \,\mathrm{d}_\tau\alpha \Big)^{1/s},$$

and since $s \ge \frac{8}{3}k + 2$, we may apply Lemma 6.4 to establish the bound

$$\int |g_j|^s \,\mathrm{d}_\tau\alpha \ll X^{s-\frac{5}{6}k-\mu},$$

for some $\mu > 0$. Consequently, one deduces that $\mathcal{J}(\mathfrak{e}) \ll X^{s-\mu+\varepsilon}$, which is acceptable.

It remains to consider the set $\mathfrak{c} \cap \mathfrak{M}(N^{1/6})$. The auxiliary estimate

$$\int_{\mathfrak{M}(N^{1/6})} \Upsilon^*(\alpha)|g(\lambda_j\alpha)|^{2t}\,\mathrm{d}_\tau\alpha \ll N^\varepsilon\Big(N^{1/6}\int_0^1 |g(\alpha)|^{2t}\,\mathrm{d}\alpha + X^{2t}\Big) \tag{6.26}$$

is now required, and this may be verified by a route paralleling that which arrives at (6.19). Note here that the irritating factor $\varphi(q)^{-1}$ can be replaced by q^{-1} at the cost of an inflationary factor $O(\log\log N)$ that can be absorbed into the term N^ε. The amplification is now similar to the work in §6.7. Let

$$\mathfrak{d}_j = \{\alpha \in \mathfrak{c} \cap \mathfrak{M}(N^{1/6}) : \lambda_j\alpha \in \mathfrak{m}(Y)\}.$$

Then, by Lemma 6.5 and Hölder's inequality, one obtains

$$\mathcal{J}(\mathfrak{d}_1) \ll X^{1-\mu} \prod_{j=2}^{2k+1} \Big(\int_{\mathfrak{d}_1} \Upsilon^*(\alpha) |g(\lambda_j\alpha)|^{2k}\, \mathrm{d}_\tau\alpha \Big)^{1/(2k)} X^{s-2k-1}.$$

On combining (6.26) with the first estimate of Lemma 6.4, it readily follows that $\mathcal{J}(\mathfrak{d}_1) \ll X^{s-\mu+\varepsilon}$. By symmetry, the same is true for $\mathcal{J}(\mathfrak{d}_j)$ when $2 \le j \le s$.

It now remains to discuss the set

$$\mathfrak{E} = \{\alpha \in \mathfrak{c} \cap \mathfrak{M}(N^{1/6}) : \lambda_j\alpha \in \mathfrak{M}(Y) \text{ for } 1 \le j \le s\}.$$

Define $T = T_{\lambda_1/\lambda_2}(\log N)$ as in (2.22) once again, and put

$$\mathfrak{D} = \{\alpha \in \mathfrak{E} : \lambda_1\alpha \in \mathfrak{M}(T^{1/4}) \text{ and } \lambda_2\alpha \in \mathfrak{M}(T^{1/4})\}.$$

Then, again as in §6.7, one finds that $\mathcal{J}(\mathfrak{D}) \ll X^s T^{-1/2}$, which is again acceptable. For $\alpha \in \mathfrak{E}\backslash\mathfrak{D}$, it follows from Lemma 6.5 that

$$g(\lambda_1\alpha)g(\lambda_2\alpha) \ll X^2 T^{\varepsilon-1/(8k)}.$$

The estimate

$$\int_{\mathfrak{E}} \Upsilon^*(\alpha)^3\, \mathrm{d}_\tau\alpha \le \int_{\mathfrak{M}(N^{1/6})} \Upsilon^*(\alpha)^3\, \mathrm{d}_\tau\alpha \ll N^2$$

follows by a simple calculation. In addition, on noting that $\frac{3}{2}(s-1) > 4k$ and $\lambda_j\alpha \in \mathfrak{M}(Y)$ for all j, an application of Hölder's inequality in alliance with Lemma 6.5 confirms the upper bound

$$\int_{\mathfrak{E}} (|g_1g_2|^{1/2}|g_3\dots g_s|)^{3/2}\, \mathrm{d}_\tau\alpha \ll X^{3(s-1)/2-k}.$$

Here, and in what follows, we have written g_i for $g(\lambda_i\alpha)$. It now follows by Hölder's inequality that

$$\int_{\mathfrak{E}\backslash\mathfrak{D}} |\Upsilon^*(\alpha) g_1\dots g_s|\, \mathrm{d}_\tau\alpha \le \sup_{\alpha\in\mathfrak{E}\backslash\mathfrak{D}} |g_1g_2|^{1/2} \Big(\int_{\mathfrak{E}} \Upsilon^*(\alpha)^3\, \mathrm{d}_\tau\alpha\Big)^{1/3} \times \Big(\int_{\mathfrak{E}} (|g_1g_2|^{1/2}|g_3\dots g_s|)^{3/2}\, \mathrm{d}_\tau\alpha\Big)^{2/3}.$$

We therefore deduce that $\mathcal{J}(\mathfrak{E}\backslash\mathfrak{D}) \ll X^s T^{\varepsilon-1/(16k)}$, a bound that in concert with our earlier estimates yields $\mathcal{J}(\mathfrak{c}) = o(X^s)$. This suffices to complete the proof of Theorem 1.9.

7. An appendix on inhomogenous polynomials

7.1. *The counting integral*

In this final chapter we sketch a proof of Theorem 1.10. Freeman's work [24] will be invoked when appropriate, but the argument relies heavily on Lemma 5.3, and is otherwise largely standard.

We keep as much notation from earlier chapters as is possible, and in particular apply the conventions of §1.5. We can then rewrite the polynomials ϕ_j as $\phi_j(t) = \lambda_{jd_j}t^{d_j} + \ldots + \lambda_{j1}t$, and rearrange the indices of $\phi_1, \ldots, \phi_s$ so as to assure that

$$\lambda_{1l_1} \textit{ and } \lambda_{2l_2} \textit{ are not in rational ratio} \tag{7.1}$$

for some $1 \le l_1 \le d_1$, $1 \le l_2 \le d_2$. To see this, suppose first that all the ϕ_j are multiples of rational polynomials. Then, there exist non-zero real numbers μ_j such that $\mu_j\phi_j \in \mathbb{Q}[t]$. Under the current hypotheses, there must be an index j with $\mu_1/\mu_j \notin \mathbb{Q}$. Exchanging j with 2, we find that (7.1) holds. In the contrary case, at least one of the polynomials ϕ_j is irrational, and we may assume that this is so for ϕ_1. Then $d_1 \ge 2$, and $\lambda_{1d_1}/\lambda_{1i}$ is irrational for some i with $1 \le i \le d_1 - 1$. Hence, one of the numbers $\lambda_{1d_1}/\lambda_{2d_2}$, $\lambda_{1i}/\lambda_{2d_2}$ is also irrational, as required.

From now on, suppose that (7.1) holds, and that ν is large. Define X_j to be the unique positive solution of $\phi_j(X_j) = \nu$. Then, for any positive solution of (1.9) with $0 < \tau \le 1$, one has $x_j \le 2X_j$. Define the Weyl sums

$$f_j(\alpha) = \sum_{x \le 2X_j} e(\alpha\phi_j(x)), \tag{7.2}$$

and the integral

$$\rho^*_{\boldsymbol{\phi}}(\tau, \nu) = \int f_1(\alpha) \ldots f_s(\alpha) e(-\nu\alpha)\, \mathrm{d}_\tau\alpha.$$

In view of (7.2) and (2.4), this is a counting integral with weight, of the type considered in (2.6), and can therefore be compared with the number $\rho_{\boldsymbol{\phi}}(\tau, \nu)$ through the now familiar mechanism based on Lemma 2.1. In particular, the proof of Theorem 1.10 is made complete with the verification of the asymptotic formula

$$\rho^*_{\boldsymbol{\phi}}(\tau, \nu) = \tfrac{1}{2}c(\boldsymbol{\phi})\tau\nu^{D-1} + o(\nu^{D-1}). \tag{7.3}$$

7.2. *The central interval*

In the interests of brevity, we put

$$\mathcal{I}(\mathfrak{a}) = \int_{\mathfrak{a}} f_1(\alpha)\dots f_s(\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha.$$

Set $d = \max d_j$, and write $Y = \nu^{1/(2d)-1}$ and $\mathfrak{C} = [-Y, Y]$. One may closely follow the proof of Lemma 2.4, or the arguments of Freeman [24], pp. 239–243, in order to evaluate $\mathcal{I}(\mathfrak{C})$. One replaces $f_j(\alpha)$ with the function

$$v_j(\alpha) = \int_0^{2X_j} e(\alpha\phi_j(t))\,\mathrm{d}t,$$

and then completes the singular integral

$$\mathcal{I}_\infty(\nu) = \int v_1(\alpha)\dots v_s(\alpha)e(-\nu\alpha)\,\mathrm{d}_\tau\alpha.$$

The error terms in these processes can be controlled by appealing to Lemma 4.4 of Baker [1] (see also Lemma 4 of [24]) and Theorem 7.3 of Vaughan [35]. In this way, one may confirm that

$$\mathcal{I}(\mathfrak{C}) = \mathcal{I}_\infty(\nu) + O(\nu^{D-1-1/(2d)}),$$

provided only that $s > 2d$. This asymptotic relation is more than we need later. Under the same condition on s, the concluding part of the proof of Lemma 2.4 is readily modified to yield

$$\mathcal{I}_\infty(\nu) = \tfrac{1}{2}c(\boldsymbol{\phi})\tau\nu^{D-1}\big(1 + O(\nu^{-1/(2d)})\big),$$

where $c(\boldsymbol{\phi})$ is the positive real number defined in the statement of Theorem 1.10. If $\mathfrak{c} = \mathbb{R}\setminus\mathfrak{C}$, then, in order to confirm (7.3), it now suffices to show that $\mathcal{I}(\mathfrak{c}) = o(\nu^{D-1})$.

7.3. *The complementary compositum*

The prearrangement in (7.1) is required only within the following lemma, which combines Lemmata 8 and 9 of Freeman [24]. We choose $\delta = 1/(2d)$ in Lemma 9 of [24], and then conclude as follows.

Lemma 7.1. *Suppose that* (7.1) *holds. Then there exists a monotone function* $T(\nu)$*, with* $T(\nu) \to \infty$ *as* $\nu \to \infty$*, such that*

$$\sup_{Y \le |\alpha| \le T(\nu)} |f_1(\alpha)f_2(\alpha)| = o(X_1X_2).$$

We remark that this is an estimate of Bentkus-Götze-Freeman type. Instead, it would be possible to work with Theorem 2.8, but at this stage the tidy reference for Lemma 7.1 saves some effort.

Now let $C > 1$ be a suitably large positive number with the property that $t(d) = d^2(\log d + 2\log\log d + C)$ is an integer. From (5.25) we have the bound

$$\int |f_j(\alpha)|^{2t(d)}\,\mathrm{d}_\tau\alpha \ll X_j^{2t(d)-d_j} \ll X_j^{2t(d)}\nu^{-1}. \tag{7.4}$$

By Lemma 11 of Freeman [24] (which is essentially already in Davenport and Roth [18], Lemma 2), it then also follows that

$$\int_{|\alpha|>T(\nu)} |f_j(\alpha)|^{2t(d)}\,\mathrm{d}_\tau\alpha \ll X_j^{2t(d)}\nu^{-1}T(\nu)^{-1}. \tag{7.5}$$

Write

$$\mathfrak{L} = \{\alpha : Y < |\alpha| < T(\nu)\} \quad \text{and} \quad \mathfrak{l} = \{\alpha : |\alpha| \geq T(\nu)\}.$$

Then for $s \geq 2t(d)+2$ and $j \geq 3$, we may apply (7.4), together with Hölder's inequality and Lemma 7.1, to confirm the estimate

$$\mathcal{I}(\mathfrak{L}) = o(X_1X_2\dots X_s\nu^{-1}).$$

Likewise, by Hölder's inequality and (7.5), we infer that

$$\mathcal{I}(\mathfrak{l}) = o(X_1X_2\dots X_s\nu^{-1}).$$

These two bounds combine to deliver the conlusion $\mathcal{I}(\mathfrak{c}) = o(\nu^{D-1})$, which was all that was required to complete the proof of Theorem 1.10.

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