ASYMPTOTIC FORMULAE FOR PAIRS OF DIAGONAL CUBIC EQUATIONS

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Abstract. We investigate the number of integral solutions possessed by a pair of diagonal cubic equations in a large box. Provided that the number of variables in the system is at least fourteen, and in addition the number of variables in any non-trivial linear combination of the underlying forms is at least eight, we obtain an asymptotic formula for the number of integral solutions consistent with the product of local densities associated with the system.

1. Introduction

The work of Davenport and Lewis [11] concerning pairs of diagonal cubic equations inaugurated the investigation of systems of additive equations in the large. Let a_i, b_i $(1 \leqslant i \leqslant s)$ be integers, and consider the pair of diophantine equations

$$a_1x_1^3 + \cdots + a_sx_s^3 = b_1x_1^3 + \cdots + b_sx_s^3 = 0. \tag{1.1}$$

Then Davenport and Lewis [11] established that the system (1.1) possesses a solution $\mathbf{x} \in \mathbb{Z}^s \setminus \{\mathbf{0}\}$ provided only that $s \geqslant 18$. The latter condition has subsequently been improved by Cook [9] to $s \geqslant 17$, and by Vaughan [15] to $s \geqslant 16$. Subject to a 7-adic solubility hypothesis, this condition was improved further by Baker and Brüdern [1] to $s \geqslant 15$, by the first author [3] to $s \geqslant 14$, and most recently by the present authors [8] to $s \geqslant 13$. Further progress would entail both negotiating arbitrarily many potential p-adic obstacles to solubility, as well as performing the miracle of extracting better than square-root cancellation on the minor arcs in an application of the circle method for cubic equations.

Amongst these opera on the topic, only Cook [9] supplies (for systems satisfying appropriate rank conditions) the expected asymptotic formula for the number $N(P) = N(P; \mathbf{a}, \mathbf{b})$ of integral solutions of the system (1.1) with $\mathbf{x} \in [-P, P]^s$. To be precise, when $s \geqslant 17$ and no coefficient ratio a_i/b_i is repeated more than $s - 9$ times for $1 \leqslant i \leqslant s$, Cook shows that

$$N(P; \mathbf{a}, \mathbf{b}) = \mathfrak{C}(\mathbf{a}, \mathbf{b})P^{s-6} + o(P^{s-6}), \tag{1.2}$$

where $\mathfrak{C}(\mathbf{a}, \mathbf{b})$ denotes the product of local densities. We note that one may write $\mathfrak{C}(\mathbf{a}, \mathbf{b})$ in the form $v_\infty \prod_p v_p$, in which v_∞ is the area of the manifold defined by (1.1) in the box $[-1, 1]^s$, and

$$v_p = \lim_{h\to\infty} p^{h(2-s)} M(p^h), \tag{1.3}$$

1991 *Mathematics Subject Classification.* 11D72, 11P55.

Key words and phrases. Exponential sums, Diophantine equations.

*The authors are grateful to the Max Planck Institute in Bonn for its generous hospitality during the period in which this paper was conceived. The second author was supported in part in the initial phases of this work by an NSF grant, and subsequently by a Royal Society Wolfson Research Merit Award.

where $M(q)$ denotes the number of solutions of the system of congruences

$$a_1x_1^3 + \cdots + a_sx_s^3 \equiv b_1x_1^3 + \cdots + b_sx_s^3 \equiv 0 \pmod q,$$

with $1 \leqslant x_i \leqslant q$ $(1 \leqslant i \leqslant s)$. Here we note that by making use of the celebrated work of Vaughan [16] concerning sums of eight cubes, it is possible to derive the anticipated asymptotic formula whenever $s \geqslant 16$, provided that no coefficient ratio is repeated more than $s - 8$ times. In the present paper we establish this asymptotic formula whenever $s \geqslant 14$, provided only that the system (1.1) satisfies appropriate rank conditions.

Before announcing our conclusions we introduce an invariant q_0 associated with the system (1.1) by defining

$$q_0 = \min_{(c,d)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}} \mathrm{card}\{1 \leqslant j \leqslant s : ca_j + db_j \neq 0\}.$$

Theorem 1.1. *Suppose that $s \geqslant 14$. Then for any choice of coefficients $(a_j, b_j) \in \mathbb{Z}^2\setminus\{\mathbf{0}\}$ $(1 \leqslant j \leqslant s)$ for which $q_0 \geqslant 8$, the asymptotic formula (1.2) holds for the simultaneous equations (1.1).*

The conclusion of Theorem 1.1 may be compared with the special case $k = 3$ of Theorem 1.1 of Brüdern and Wooley [6], wherein a similar result is obtained with $s \geqslant 15$ in place of the above condition $s \geqslant 14$. When $s \geqslant 13$, meanwhile, the principal conclusions of [7] and [8] supply the lower bound $N(P;\mathbf{a},\mathbf{b}) \gg \mathfrak{C}(\mathbf{a},\mathbf{b})P^{s-6}$, of the anticipated order of magnitude, whenever $q_0 \geqslant 7$. As already noted above, the earlier work of Cook, as modified by Vaughan's methods, is subject to the more onerous constraint $s \geqslant 16$. We remark also that by considering equations containing disjoint sets of variables, one finds that the hypotheses of the theorem concerning the invariant q_0 of the system cannot be weakened without first establishing the validity of the anticipated asymptotic formula for a single diagonal cubic equation in fewer than 8 variables. Finally, we should point out that work of Cook [10] shows for $s \geqslant 13$ that the system (1.1) possesses non-trivial p-adic solutions whenever $p \neq 7$, and that earlier work of Davenport and Lewis [11] establishes such for every prime p provided that $s \geqslant 16$. In the circumstances dictated by the hypotheses of Theorem 1.1, it follows that the product of local densities $\mathfrak{C}(\mathbf{a},\mathbf{b})$ is bounded away from zero subject only to the 7-adic solubility of the system (1.1).

Our proof of Theorem 1.1 makes use of a mean value estimate for a fourteenth moment of Weyl sums. Subject to rank conditions on the underlying coefficient matrix, recent work of the authors [4], [6] establishes such an estimate that would miss a variant of Theorem 1.1 by only ε in the exponent, with $\varepsilon > 0$ arbitrarily small. By carefully modifying the argument applied in our recent work on paucity problems [5], we are able to make additional savings large enough to obtain an estimate of sufficient strength to establish Theorem 1.1. This strategy involves bounding in mean square certain Fourier coefficients restricted to a thin set. We discuss such matters in §2 below. With this mean value estimate in hand, and some routine preparation in §3, the proof of Theorem 1.1 is a fairly routine application of the Hardy-Littlewood method that we bring to completion in §4.

Throughout, the letter ε will denote a sufficiently small positive number, and P will be a large real number. We use $\ll$ and $\gg$ to denote Vinogradov's notation. In an effort to simplify our account, whenever ε appears in a statement, we assert that the statement holds for every positive number ε. The "value" of ε may consequently change from statement to statement.

2. Restricted mean square estimates for Fourier coefficients

In this section we establish an estimate for a certain fourteenth moment of cubic Weyl sums, our strategy being to interpret this moment as a restricted mean square of Fourier coefficients. We consider a real-valued function F in $L^2([0,1))$, extended in the natural way to a periodic function on $\mathbb{R}$ with period 1. When $\mathfrak{B} \subseteq [0,1)$ is measurable, we put

$$R(n;\mathfrak{B}) = \int_{\mathfrak{B}} F(\gamma)e(-n\gamma)d\gamma,$$

where, as usual, we write $e(z)$ for $e^{2\pi iz}$. Also, we denote by $\rho(n)$ the number of representations of the integer n in the shape $n = x^3 - y^3$, with $|x|,|y| \leqslant P$. It is convenient then to write $\mathcal{C}$ for the set of integers n, with $1 \leqslant |n| \leqslant 2P^3$, for which $\rho(n) > 2$, so that n necessarily possesses more than one essentially distinct representation in the aforementioned manner. In addition, we put $\mathcal{C}_0 = \mathcal{C} \cup \{0\}$.

Lemma 2.1. *Whenever h is a fixed non-zero integer and $\mathfrak{B} \subseteq [0,1)$ is measurable, one has*

$$\sum_{n\in\mathbb{Z}} \rho(n)|R(hn;\mathfrak{B})|^2 \ll P\left(\int_{\mathfrak{B}} F(\gamma)\,d\gamma\right)^2 + \int_{\mathfrak{B}} F(\gamma)^2\,d\gamma + P^\varepsilon \sum_{n\in\mathcal{C}} |R(hn;\mathfrak{B})|^2. \qquad (2.1)$$

Proof. On noting that $\rho(0) = O(P)$, one finds that the contribution to the sum on the left hand side of (2.1) arising from the term $n = 0$, say T_0, satisfies

$$T_0 \ll P|R(0;\mathfrak{B})|^2 = P\left(\int_{\mathfrak{B}} F(\gamma)\,d\gamma\right)^2. \qquad (2.2)$$

When $n \in \mathbb{Z}\backslash\mathcal{C}_0$, meanwhile, we have $\rho(n) = O(1)$, and thus we find that the contribution, T_1, of such terms within (2.1) satisfies

$$T_1 \ll \sum_{n\in\mathbb{Z}\backslash\mathcal{C}_0} |R(hn;\mathfrak{B})|^2 \leqslant \sum_{m\in\mathbb{Z}} |R(m;\mathfrak{B})|^2.$$

An application of Bessel's inequality therefore reveals that

$$T_1 \ll \int_{\mathfrak{B}} F(\gamma)^2\,d\gamma. \qquad (2.3)$$

Finally, when $n \neq 0$, a divisor function estimate yields $\rho(n) = O(|n|^\varepsilon)$, and so the contribution arising from the terms with $n \in \mathcal{C}$, say T_2, satisfies

$$T_2 \ll P^\varepsilon \sum_{n\in\mathcal{C}} |R(hn;\mathfrak{B})|^2. \qquad (2.4)$$

The conclusion of the lemma now follows by summing the estimates (2.2), (2.3) and (2.4). □

When P is a large positive number, let

$$f(\gamma) = \sum_{|x|\leqslant P} e(\gamma x^3). \qquad (2.5)$$

In our application of Lemma 2.1 that establishes the promised mean value estimate, we take $F(\gamma) = |f(l\gamma)|^6$ for a fixed non-zero integer l. It transpires that the first two terms on the right hand side of (2.1) are well-controlled, but that the third term has the potential to be larger than the desired bound by a factor of P^ε. We therefore concentrate on this last term, and apply slim-stout technology in order to bound its contribution. In order to facilitate this discussion, when $l \in \mathbb{Z}$ and $\mathfrak{B} \subseteq [0,1)$ is measurable, we now put

$$\psi_l(n;\mathfrak{B}) = \int_{\mathfrak{B}} |f(l\gamma)|^6 e(-n\gamma)\,d\gamma.$$

Also, given a positive number Q with $1 \leqslant Q \leqslant P$, we define two sets of major arcs $\mathfrak{M}(Q)$ and $\mathfrak{N}(Q)$ to be the respective unions of the intervals

$$\mathfrak{M}(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant QP^{-3}\}$$

and

$$\mathfrak{N}(q,a) = \{\alpha \in [0,1) : |\alpha - a/q| \leqslant QP^{-3}\},$$

with $0 \leqslant a \leqslant q \leqslant Q$ and $(a,q)=1$. We then put $\mathfrak{m}(Q) = [0,1) \setminus \mathfrak{M}(Q)$ and $\mathfrak{n}(Q) = [0,1) \setminus \mathfrak{N}(Q)$.

In advance of a lemma that provides a mean square estimate for $\psi_l(n;\mathfrak{B})$, it is convenient to record an elementary property of the sets $\mathfrak{M}(Q)$, $\mathfrak{m}(Q)$, $\mathfrak{N}(Q)$ and $\mathfrak{n}(Q)$ for differing values of Q.

Lemma 2.2. *Suppose that Q is a positive number with $1 \leqslant Q \leqslant P$ and that l is a fixed positive integer. Then the following hold.*

(a) Whenever α is a real number with $\alpha \in \mathfrak{M}(Q)$ modulo 1, one has $l\alpha \in \mathfrak{M}(lQ)$ modulo 1 and, for each integer k, one has $(\alpha+k)/l \in \mathfrak{M}(lQ)$ modulo 1.

(b) Whenever α is a real number with $\alpha \in \mathfrak{m}(Q)$ modulo 1, one has $l\alpha \in \mathfrak{m}(Q/l)$ modulo 1 and, for each integer k, one has $(\alpha+k)/l \in \mathfrak{m}(Q/l)$ modulo 1.

The same conclusions hold when $\mathfrak{M}$ is replaced by $\mathfrak{N}$ throughout, and likewise when $\mathfrak{m}$ is replaced by $\mathfrak{n}$.

Proof. The proof of the lemma in the situation in which $\mathfrak{N}$ replaces $\mathfrak{M}$ in statement (a), and $\mathfrak{n}$ replaces $\mathfrak{m}$ in statement (b), is a trivial modification of the argument that follows. We therefore concentrate on the explicitly recorded assertions and leave the modifications necessary to address the sets $\mathfrak{N}$ and $\mathfrak{n}$ to the reader.

Suppose first that α is a real number with $\alpha \in \mathfrak{M}(Q)$ modulo 1. Then there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a,q)=1$, $q \leqslant Q$ and $|q\alpha - a| \leqslant QP^{-3}$. Consequently, one has

$$|q(l\alpha) - al| \leqslant lQP^{-3} \quad \text{with} \quad q \leqslant Q,$$

whence $l\alpha \in \mathfrak{M}(lQ)$ modulo 1. In addition, for each integer k one has

$$\left| ql\left(\frac{\alpha+k}{l}\right) - (qk+a)\right| = |q\alpha - a| \leqslant QP^{-3} \quad \text{with} \quad ql \leqslant lQ,$$

whence $(\alpha+k)/l \in \mathfrak{M}(lQ)$ modulo 1. This confirms the assertions of part (a) of the lemma.

Next, if α is a real number for which either $l\alpha \in \mathfrak{M}(Q/l)$ modulo 1, or for some integer k one has $(\alpha+k)/l \in \mathfrak{M}(Q/l)$ modulo 1, it follows from part (a) that $\alpha \in \mathfrak{M}(Q)$ modulo 1. The assertion of part (b) consequently follows from the observation that $\mathfrak{m}(Q)$ is the complement of $\mathfrak{M}(Q)$ within $[0,1)$. $\square$

For the sake of concision, we now abbreviate $\mathfrak{M}(P^{3/4})$ to $\mathfrak{M}_1$ and likewise $\mathfrak{m}(P^{3/4})$ to $\mathfrak{m}_1$.

Lemma 2.3. *Suppose that h and l are fixed non-zero integers. Then whenever $\mathfrak{B} \subseteq \mathfrak{m}_1$ is measurable, one has*

$$\sum_{n \in \mathcal{C}} |\psi_l(hn;\mathfrak{B})|^2 \ll P^{47/6+\varepsilon}.$$

Proof. When T is a non-negative real number, we write $\mathcal{Z}(T)$ for the set of integers $n \in \mathcal{C}$ for which $T < |\psi_l(hn;\mathfrak{B})| \leqslant 2T$, and then abbreviate $\mathrm{card}(\mathcal{Z}(T))$ to $Z(T)$. Also, when $n \in \mathcal{Z}(T)$ we define $\sigma(n)$ by means of the relation $|\psi_l(hn;\mathfrak{B})| = \sigma(n)\psi_l(hn;\mathfrak{B})$, and then put

$$K_T(\alpha) = \sum_{n \in \mathcal{Z}(T)} \sigma(n) e(hn\alpha).$$

It follows that

$$\begin{aligned}\int_{\mathfrak{B}} |f(l\gamma)|^6 K_T(-\gamma)\, d\gamma &= \sum_{n \in \mathcal{Z}(T)} \sigma(n) \int_{\mathfrak{B}} |f(l\gamma)|^6 e(-hn\gamma)\, d\gamma \\ &= \sum_{n \in \mathcal{Z}(T)} \sigma(n)\psi_l(hn;\mathfrak{B}) = \sum_{n \in \mathcal{Z}(T)} |\psi_l(hn;\mathfrak{B})| \\ &> TZ(T).\end{aligned}$$

By Schwarz's inequality, one therefore deduces that

$$TZ(T) < I_1^{1/2} I_2^{1/2}, \tag{2.6}$$

where

$$I_1 = \int_0^1 |f(l\gamma)K_T(\gamma)|^2\, d\gamma \quad \text{and} \quad I_2 = \int_{\mathfrak{B}} |f(l\gamma)|^{10} d\gamma.$$

By orthogonality, the integral I_1 is bounded above by the number of solutions of the diophantine equation $l(x_1^3 - x_2^3) = h(n_1 - n_2)$, with $|x_1|, |x_2| \leqslant P$ and $n_1, n_2 \in \mathcal{Z}(T)$. The number of diagonal solutions of this equation, wherein $x_1 = x_2$ and $n_1 = n_2$, is plainly $O(PZ(T))$. For each of the $O(Z(T)^2)$ fixed choices of n_1 and n_2 with $n_1 \neq n_2$, meanwhile, it follows from a divisor function estimate that the number of choices for x_1 and x_2, with $l(x_1^3 - x_2^3)$ equal to the fixed non-zero integer $h(n_1 - n_2)$, is $O(P^\varepsilon)$. Thus we see that

$$I_1 \ll PZ(T) + P^\varepsilon Z(T)^2. \tag{2.7}$$

With a future application in mind, we estimate I_2 with greater precision than is warranted by the purpose at hand. Observe first that by combining the refined estimates of Hall and Tenenbaum [12] for Hooley's Δ-function with the proof of Lemma 1 of Vaughan [16], one has the estimate

$$\sup_{\gamma \in \mathfrak{B}} |f(l\gamma)| \leqslant \sup_{\gamma \in \mathfrak{m}_1} |f(l\gamma)| \ll \sup_{\gamma \in \mathfrak{m}(l^{-1}P^{3/4})} |f(\gamma)| \ll P^{3/4}(\log P)^{1/4+\varepsilon}. \tag{2.8}$$

Here we have made use of Lemma 2.2(b) in order to justify the second of this string of asymptotic inequalities. Next, by referring to Boklan [2], a second application of Lemma 2.2(b) yields the upper bound

$$\int_{\mathfrak{B}} |f(l\gamma)|^8\, d\gamma \leqslant \int_{\mathfrak{m}_1} |f(l\gamma)|^8\, d\gamma \ll \int_{\mathfrak{m}(l^{-1}P^{3/4})} |f(\gamma)|^8\, d\gamma \ll P^5(\log P)^{\varepsilon-3}. \tag{2.9}$$

The estimates (2.8) and (2.9) together imply that

$$I_2 \leqslant \Big(\sup_{\gamma\in\mathfrak{B}} |f(l\gamma)|\Big)^2 \int_{\mathfrak{B}} |f(l\gamma)|^8\,d\gamma \ll P^{13/2}. \tag{2.10}$$

On substituting (2.7) and (2.10) into (2.6), we arrive at the estimate

$$TZ(T) \ll P^\varepsilon (PZ(T)+Z(T)^2)^{1/2}(P^{13/2})^{1/2}.$$

Given any fixed positive number δ, therefore, it follows that for $T > P^{13/4+\delta}$ one has $Z(T) \ll P^{15/2+\varepsilon}T^{-2}$. On summing over the contributions arising from all available intervals of the type $T < |\psi_l(hn;\mathfrak{B})| \leqslant 2T$ with $T > P^{13/4+\delta}$, we find that

$$\sum_{\substack{n\in\mathcal{C}\\ |\psi_l(hn;\mathfrak{B})|>P^{13/4+\delta}}} |\psi_l(hn;\mathfrak{B})|^2 \ll P^\varepsilon \max_{P^{13/4+\delta}<T\leqslant (2P+1)^6} \big(T^2 Z(T)\big) \ll P^{15/2+\varepsilon}. \tag{2.11}$$

But it follows from Theorem 1 of Heath-Brown [13] that $\mathrm{card}(\mathcal{C}) = O(P^{4/3+\varepsilon})$ (see the discussion leading to the estimate (9.10) of Wooley [19] for an account of this estimate). We may therefore infer that

$$\sum_{\substack{n\in\mathcal{C}\\ |\psi_l(hn;\mathfrak{B})|\leqslant P^{13/4+\delta}}} |\psi_l(hn;\mathfrak{B})|^2 \ll P^{4/3+\varepsilon}(P^{13/4+\delta})^2 \ll P^{47/6+3\delta}. \tag{2.12}$$

The conclusion of the lemma is now immediate from (2.11) and (2.12). □

We are now equipped to establish the mean square estimate for certain Fourier coefficients that is crucial in our proof of the promised fourteenth moment of exponential sums. In this context, when λ is a fixed positive number, we now write $\mathfrak{M}_{2,\lambda}$ for $\mathfrak{M}(\lambda(\log P)^3)$ and $\mathfrak{m}_{2,\lambda}$ for $\mathfrak{m}(\lambda(\log P)^3)$, and to save space we abbreviate $\mathfrak{M}_1\setminus\mathfrak{M}_{2,\lambda}$ to $\mathfrak{P}_\lambda$.

Lemma 2.4. *Suppose that h and l are fixed non-zero integers, and that λ is a fixed positive number. Then whenever $\mathfrak{B}\subseteq\mathfrak{m}_{2,\lambda}$ is measurable, one has*

$$\sum_{n\in\mathbb{Z}} \rho(n)|\psi_l(hn;\mathfrak{B})|^2 \ll P^8(\log P)^{\varepsilon-2}.$$

Proof. Since $\mathfrak{m}_{2,\lambda}$ is the union of $\mathfrak{m}_1$ and $\mathfrak{M}_1\setminus\mathfrak{M}_{2,\lambda}$, and $\mathfrak{B}\subseteq\mathfrak{m}_{2,\lambda}$, we see that

$$\sum_{n\in\mathbb{Z}} \rho(n)|\psi_l(hn;\mathfrak{B})|^2 \ll \sum_{n\in\mathbb{Z}} \rho(n)|\psi_l(hn;\mathfrak{m}_1)|^2 + \sum_{n\in\mathbb{Z}} \rho(n)|\psi_l(hn;\mathfrak{P}_\lambda)|^2. \tag{2.13}$$

The estimation of the first term on the right hand side of (2.13) is completed swiftly via Lemma 2.1, as we now demonstrate. We set $|F(\gamma)| = |f(l\gamma)|^6$ in our application of the latter lemma, and begin by noting that from (2.8) and (2.9) one has

$$\int_{\mathfrak{m}_1} |f(l\gamma)|^{12}\,d\gamma \leqslant \Big(\sup_{\gamma\in\mathfrak{m}_1} |f(l\gamma)|\Big)^4 \int_{\mathfrak{m}_1} |f(l\gamma)|^8\,d\gamma \ll P^8(\log P)^{\varepsilon-2}. \tag{2.14}$$

Next, on recalling the estimate

$$\int_0^1 |f(l\gamma)|^4\,d\gamma \ll P^2,$$

which, on considering the underlying diophantine equation, is an immediate consequence of Theorem 3 of Hooley [14], we deduce from (2.9) in combination with Schwarz's inequality that

$$\int_{\mathfrak{m}_1} |f(l\gamma)|^6 \, d\gamma \leqslant \left(\int_0^1 |f(l\gamma)|^4 \, d\gamma \right)^{1/2} \left(\int_{\mathfrak{m}_1} |f(l\gamma)|^8 \, d\gamma \right)^{1/2} \ll P^{7/2} (\log P)^{\varepsilon - 3/2}. \tag{2.15}$$

Applying (2.14) and (2.15) in combination with Lemma 2.3 within Lemma 2.1, we conclude that

$$\sum_{n \in \mathbb{Z}} \rho(n) |\psi_l(hn; \mathfrak{m}_1)|^2 \ll P \left(P^{7/2} (\log P)^{\varepsilon - 3/2} \right)^2 + P^8 (\log P)^{\varepsilon - 2} + P^{47/6 + \varepsilon} \ll P^8 (\log P)^{\varepsilon - 2}. \tag{2.16}$$

The second term on the right hand side of (2.13) may be estimated using major arc technology familiar to modern practitioners of the circle method. First, an application of Lemma 5.1 of Vaughan [17] in combination with the triangle inequality confirms that whenever $|n| \leqslant P^3$, one has

$$|\psi_l(hn; \mathfrak{P}_\lambda)| \leqslant \int_{\mathfrak{M}_1 \setminus \mathfrak{M}_{2,\lambda}} |f(l\gamma)|^6 \, d\gamma \ll P^3 (\log P)^{\varepsilon - 1}.$$

Next, on recalling the definition of $\rho(n)$, we deduce that

$$\sum_{n \in \mathbb{Z}} \rho(n) |\psi_l(hn; \mathfrak{P}_\lambda)|^2 \ll (P^3 (\log P)^{\varepsilon - 1})^2 \sum_{n \in \mathbb{Z}} \rho(n) \ll P^8 (\log P)^{2\varepsilon - 2}.$$

The proof of the lemma is now completed by substituting the latter estimate together with (2.16) into (2.13). □

We are at last prepared to establish the fundamental mean value estimate of this paper. In this context we define a two dimensional Hardy-Littlewood dissection as follows. Given a positive number Q with $1 \leqslant Q \leqslant P$, we define the set of major arcs $\mathfrak{W}(Q)$ to be the union of the boxes

$$\mathfrak{W}(q, a, b) = \{ (\alpha, \beta) \in [0,1)^2 \, : \, |\alpha - a/q| \leqslant QP^{-3} \text{ and } |\beta - b/q| \leqslant QP^{-3} \},$$

with $0 \leqslant a, b \leqslant q \leqslant Q$ and $(q, a, b) = 1$. We also define the set of minor arcs $\mathfrak{w}(Q)$ by putting $\mathfrak{w}(Q) = [0,1)^2 \setminus \mathfrak{W}(Q)$. We remark for future reference that $\mathfrak{N}(Q^{1/2}) \times \mathfrak{N}(Q^{1/2}) \subseteq \mathfrak{W}(Q)$, as is easily verified, and thus

$$\mathfrak{w}(Q) \subseteq \left(\mathfrak{n}(Q^{1/2}) \times \mathfrak{n}(Q^{1/2}) \right) \cup \left(\mathfrak{n}(Q^{1/2}) \times \mathfrak{N}(Q^{1/2}) \right) \cup \left(\mathfrak{N}(Q^{1/2}) \times \mathfrak{n}(Q^{1/2}) \right). \tag{2.17}$$

It is convenient in what follows to define the linear forms Λ_j by

$$\Lambda_j = a_j \alpha + b_j \beta \quad (1 \leqslant j \leqslant s). \tag{2.18}$$

In addition, we put $X = (\log P)^6$ and abbreviate $\mathfrak{w}(X)$ to $\mathfrak{w}$, and $\mathfrak{W}(X)$ to $\mathfrak{W}$.

Theorem 2.5. *Suppose that Λ_u, Λ_v and Λ_w are pairwise linearly independent linear forms in α and β. Then one has*

$$\iint_{\mathfrak{w}} |f(\Lambda_u)^6 f(\Lambda_v)^6 f(\Lambda_w)^2| \, d\alpha \, d\beta \ll P^8 (\log P)^{\varepsilon - 1}. \tag{2.19}$$

Proof. We begin by investigating the change of variables $(\alpha,\beta)\mapsto(\Lambda_u,\Lambda_v)$, and its effect on the range of integration $\mathfrak{w}(X)$. Consider then a positive parameter Q, and the linear combinations $a_v\Lambda_u-a_u\Lambda_v$ and $b_v\Lambda_u-b_u\Lambda_v$. Define

$$\Omega'=\max_{1\leqslant i\leqslant j\leqslant s}\{|a_i|+|a_j|\,,\,|b_i|+|b_j|\},$$

write $C=a_ub_v-a_vb_u$, and put $\Omega=\Omega'|C|$. Note here that the hypotheses of the lemma ensure that Ω is non-zero. One may check that whenever $(\Lambda_u,\Lambda_v)\in\mathfrak{W}(\Omega^{-1}Q)$ modulo 1, then necessarily $C(\alpha,\beta)\in\mathfrak{W}(\Omega'\Omega^{-1}Q)$ modulo 1. But in such circumstances, an argument paralleling that of the proof of part (a) of Lemma 2.2 shows that $(\alpha,\beta)\in\mathfrak{W}(|C|\Omega'\Omega^{-1}Q)=\mathfrak{W}(Q)$. Consequently, whenever $(\alpha,\beta)\in\mathfrak{w}(Q)$ modulo 1, one necessarily has $(\Lambda_u,\Lambda_v)\in\mathfrak{w}(\Omega^{-1}Q)$ modulo 1. Furthermore, whenever $(\Lambda_u,\Lambda_v)\in\mathfrak{w}(\Omega^{-1}Q)$ modulo 1, one has $C^{-1}(\Lambda_u,\Lambda_v)\in\mathfrak{w}(|C|^{-1}\Omega^{-1}Q)$ modulo 1. We note also that on writing $A=a_wb_v-a_vb_w$ and $B=a_ub_w-a_wb_u$, one has $C\Lambda_w=A\Lambda_u+B\Lambda_v$. Here, in view of the hypotheses of the lemma, neither A nor B is zero.

We now write

$$\mathfrak{u}=\mathfrak{w}(\Omega^{-1}X)\quad\text{and}\quad\mathfrak{v}=\mathfrak{w}(|C|^{-1}\Omega^{-1}X).$$

Then in view of the above discussion, we may make use of the periodicity (with period 1) of the integrand on the left hand side of (2.19), and change variables, so as to obtain the relation

$$\iint_{\mathfrak{w}}|f(\Lambda_u)^6f(\Lambda_v)^6f(\Lambda_w)^2|\,d\alpha\,d\beta \ll\iint_{\mathfrak{u}}|f(\Lambda_u)^6f(\Lambda_v)^6f(C^{-1}(A\Lambda_u+B\Lambda_v))^2|\,d\Lambda_u\,d\Lambda_v.$$

On putting $\xi=C^{-1}\Lambda_u$ and $\eta=C^{-1}\Lambda_v$, and making a second change of variables, we arrive at the estimate

$$\iint_{\mathfrak{u}}|f(\Lambda_u)^6f(\Lambda_v)^6f(C^{-1}(A\Lambda_u+B\Lambda_v))^2|\,d\Lambda_ud\Lambda_v \ll\iint_{\mathfrak{v}}|f(C\xi)^6f(C\eta)^6f(A\xi+B\eta)^2|\,d\xi\,d\eta.$$

Put

$$\mathfrak{X}=\mathfrak{N}((\Omega|C|)^{-1/2}X^{1/2})\quad\text{and}\quad\mathfrak{x}=\mathfrak{n}((\Omega|C|)^{-1/2}X^{1/2}).$$

Then on recalling (2.17), we deduce that

$$\iint_{\mathfrak{w}}|f(\Lambda_u)^6f(\Lambda_v)^6f(\Lambda_w)^2|\,d\alpha\,d\beta\ll I(\mathfrak{x},\mathfrak{x})+I(\mathfrak{x},\mathfrak{X})+I(\mathfrak{X},\mathfrak{x}),\tag{2.20}$$

where we have written

$$I(\mathcal{A},\mathcal{B})=\int_{\mathcal{A}}\int_{\mathcal{B}}|f(C\xi)^6f(C\eta)^6f(A\xi+B\eta)^2|\,d\xi\,d\eta.$$

We enter the final phase of the proof by rewriting the integral $I(\mathcal{A},\mathcal{B})$ in a form amenable to Lemma 2.4. We observe that

$$I(\mathcal{A},\mathcal{B}) = \sum_{|x|\leqslant P}\sum_{|y|\leqslant P}\int_{\mathcal{A}} |f(C\xi)|^6 e(-A(x^3-y^3)\xi)\,d\xi \int_{\mathcal{B}} |f(C\eta)|^6 e(-B(x^3-y^3)\eta)\,d\eta$$
$$= \sum_{n\in\mathbb{Z}} \rho(n)\psi_C(An;\mathcal{A})\psi_C(Bn;\mathcal{B}).$$

An application of Cauchy's inequality now reveals that

$$I(\mathcal{A},\mathcal{B}) \leqslant J_{C,A}(\mathcal{A})^{1/2} J_{C,B}(\mathcal{B})^{1/2}, \tag{2.21}$$

where

$$J_{S,T}(\mathcal{C}) = \sum_{n\in\mathbb{Z}} \rho(n)|\psi_S(Tn;\mathcal{C})|^2.$$

On recalling that $X=(\log P)^6$, we see that $\mathfrak{x}=\mathfrak{m}_{2,\lambda}$ with $\lambda=(\Omega|C|)^{-1/2}$, and thus it follows from Lemma 2.4 that $J_{C,T}(\mathfrak{x}) = O(P^8(\log P)^{\varepsilon-2})$ for $T=A,B$. Meanwhile, the estimate $\psi_C(Tn;\mathfrak{X}) = O(P^3)$ follows from the methods underlying the proof of Lemma 5.1 of Vaughan [17]. Thus, again recalling the definition of $\rho(n)$, one finds that for $T=A,B$, one has

$$J_{C,T}(\mathfrak{X}) \ll (P^3)^2 \sum_{n\in\mathbb{Z}} \rho(n) \ll P^8.$$

It therefore follows from (2.20) and (2.21) that

$$\int_{\mathfrak{w}} |f(\Lambda_u)^6 f(\Lambda_v)^6 f(\Lambda_w)^2|\,d\alpha\,d\beta \ll (P^8(\log P)^{\varepsilon-2})^{1/2}(P^8)^{1/2},$$

and the conclusion of the theorem is now immediate. □

3. Preparation for the circle method

Our purpose in this and the next section of this paper is to prove Theorem 1.1. In view of the hypotheses of the theorem, we may suppose henceforth that $q_0 \geqslant 8$. With the coefficient pairs $(a_j,b_j)\in\mathbb{Z}^2\setminus\{\mathbf{0}\}$ $(1\leqslant j\leqslant s)$, we associate both the linear forms Λ_j defined in (2.18) and the two linear forms $L_1(\boldsymbol{\theta})$ and $L_2(\boldsymbol{\theta})$ defined for $\boldsymbol{\theta}\in\mathbb{R}^s$ by

$$L_1(\boldsymbol{\theta}) = \sum_{j=1}^s a_j\theta_j \quad\text{and}\quad L_2(\boldsymbol{\theta}) = \sum_{j=1}^s b_j\theta_j. \tag{3.1}$$

We say that two forms Λ_i and Λ_j are *equivalent* when there exists a non-zero rational number λ with $\Lambda_i=\lambda\Lambda_j$. This notion defines an equivalence relation on the set $\{\Lambda_1,\Lambda_2,\dots,\Lambda_s\}$, and we refer to the number of elements in the equivalence class $[\Lambda_j]$ containing the form Λ_j as its *multiplicity*. Suppose that the s forms Λ_j $(1\leqslant j\leqslant s)$ fall into T equivalence classes, and that the multiplicities of these classes are $R_1,R_2,\dots,R_T$. By relabelling variables if necessary, there is no loss in supposing that $R_1\geqslant R_2\geqslant\ldots\geqslant R_T\geqslant 1$. Further, in view of our hypothesis that $q_0\geqslant 8$, it is apparent that we may assume without loss of generality that $R_1\leqslant s-8$, and hence that $R_2+R_3+\cdots+R_T\geqslant 8$.

We distinguish three cases according to the number of variables and the arrangement of the multiplicities of the forms. We refer to a system (1.1) as being of type I when $T=2$, as being of type II when $T=3$ and $R_3=1$, and as being of type III in the remaining cases wherein $T\geqslant 3$ and $s-R_1-R_2\geqslant 2$. We defer the treatment of systems of type I to the final paragraph of §4 below, confining our attention in the remainder of §3 to systems of types II and III.

Suppose first that the system (1.1) is of type III with $s \geqslant 14$ and $q_0 \geqslant 8$. We consider fixed subsets $\mathcal{S}_0$ and $\mathcal{S}_1$ of $\{1,\ldots,s\}$ with $\mathrm{card}(\mathcal{S}_0)=13$, $\mathrm{card}(\mathcal{S}_1)=14$ and $\mathcal{S}_0 \subset \mathcal{S}_1$. We may suppose that the 14 forms Λ_j $(j \in \mathcal{S}_1)$ fall into t equivalence classes, and that the multiplicities of the representatives of these classes are $r_1,\ldots,r_t$. By relabelling variables if necessary, there is no loss in supposing that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_t \geqslant 1$. Then on recalling the additional conditions $s \geqslant 14$, $T \geqslant 3$ and $s-R_1-R_2 \geqslant 2$, we may plainly make a choice for $\mathcal{S}_1$ in such a manner that $t \geqslant 3$, $1 \leqslant r_1 \leqslant 6$ and $14-r_1-r_2 \geqslant 2$. Furthermore, it is trivially apparent that we may choose the subset $\mathcal{S}_0$ of $\mathcal{S}_1$ in such a manner that the maximum multiplicity of the representatives of the 13 forms Λ_j $(j \in \mathcal{S}_0)$ is at most six amongst the latter forms. In view of our earlier condition $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_t \geqslant 1$, moreover, one has also the constraint $r_t \leqslant 4$.

Suppose next that the system (1.1) is of type II with $s \geqslant 14$ and $q_0 \geqslant 8$. Then one has $T=3$ and $R_3=1$, and so the hypothesis $q_0 \geqslant 8$ implies that one necessarily has $R_1 \geqslant R_2 \geqslant 7$ and $s \geqslant 15$. In this case we consider fixed subsets $\mathcal{S}_0$ and $\mathcal{S}_2$ of $\{1,\ldots,s\}$ with $\mathrm{card}(\mathcal{S}_0)=13$, $\mathrm{card}(\mathcal{S}_2)=15$ and $\mathcal{S}_0 \subset \mathcal{S}_2$. We may suppose that the 15 forms Λ_j $(j \in \mathcal{S}_2)$ fall into 3 equivalence classes, and that the multiplicities of the representatives of these classes are r_1, r_2, r_3. By relabelling variables if necessary, there is no loss in supposing that $(r_1,r_2,r_3)=(7,7,1)$. In particular, we may make a subsequent choice for $\mathcal{S}_0 \subset \mathcal{S}_2$ so that the maximum multiplicity of the representatives of the 13 forms Λ_j $(j \in \mathcal{S}_0)$ is again at most six amongst the latter forms.

In each of the above situations, one may relabel variables in the system (1.1), and likewise in (2.18) and (3.1), so that the set $\mathcal{S}_l$ becomes $\{1,2,\ldots,13+l\}$ $(l=0,1,2)$, and so that Λ_1 becomes a form in the first equivalence class counted by r_1, and Λ_2 becomes a form in the second equivalence class counted by r_2. Some simplifying transformations ease the analysis of the singular integral, and here we follow the pattern of our earlier work [7], [8]. First, by taking suitable integral linear combinations of the equations (1.1), we may suppose without loss that $b_1=a_2=0$. Since we may suppose that $a_1b_2 \neq 0$, the simultaneous equations

$$L_1(\boldsymbol{\theta})=L_2(\boldsymbol{\theta})=0 \tag{3.2}$$

possess a solution $\boldsymbol{\theta}$ with $\theta_j \neq 0$ $(1 \leqslant j \leqslant s)$. Applying the substitution $x_j \to -x_j$ for those indices j with $1 \leqslant j \leqslant s$ for which $\theta_j<0$, neither the solubility of the system (1.1) nor the corresponding function $N(P)$, are affected, and yet the transformed linear system associated with the equations (3.2) has a solution $\boldsymbol{\theta}$ with $\theta_j>0$ $(1 \leqslant j \leqslant s)$. In addition, the homogeneity of the system (3.2) ensures that a solution of the latter type may be chosen with $\boldsymbol{\theta} \in (0,1)^s$. We now fix the solution $\boldsymbol{\theta}$.

We next introduce some additional generating functions required in our application of the circle method. We employ the classical Weyl sum $f(\alpha)$ defined already in (2.5). In addition, for each natural number m with $m \leqslant s$, we define the generating functions

$$H_m(\alpha,\beta)=\prod_{j=1}^{m} f(\Lambda_j) \quad \text{and} \quad G_m(\alpha,\beta)=\prod_{j=m+1}^{s} f(\Lambda_j). \tag{3.3}$$

Here, in circumstances in which $m=s$, the last product is empty and interpreted as unity. We note the trivial decomposition $H_s(\alpha,\beta)=H_m(\alpha,\beta)G_m(\alpha,\beta)$ $(m \leqslant s)$. From orthogonality we now have the relation

$$N(P;\mathbf{a},\mathbf{b})=\int_0^1\int_0^1 H_s(\alpha,\beta)\,d\alpha\,d\beta. \tag{3.4}$$

The contribution of the major arcs within the integral on the right hand side of (3.4) is easily estimated by making use of work from our previous paper [8].

Lemma 3.1. *Suppose that the system (1.1) is of type* II *or* III *with* $s\geqslant 13$ *and* $q_0\geqslant 7$. *Then one has*

$$\iint_{\mathfrak{W}} H_s(\alpha,\beta)\,d\alpha\,d\beta=\mathfrak{C}(\mathbf{a},\mathbf{b})P^{s-6}+o(P^{s-6}).$$

Moreover, provided that the system (1.1) *possesses a non-trivial* 7*-adic solution, one has* $\mathfrak{C}(\mathbf{a},\mathbf{b})\gg 1$.

Proof. We begin by introducing some notation in order to discuss the approximant to the generating function f on the major arcs $\mathfrak{W}$. Let

$$S(q,r)=\sum_{l=1}^q e(rl^3/q)\quad\text{and}\quad S_i(q,c,d)=S(q,a_ic+b_id)\quad(1\leqslant i\leqslant s).$$

Also, for $1\leqslant j\leqslant s$ put $\lambda_j=a_j\xi+b_j\zeta$, and write $w_j(\xi,\zeta)$ for $w(\lambda_j)$, where

$$w(\theta)=\int_{-P}^P e(\theta\gamma^3)\,d\gamma.$$

As a consequence of Theorem 4.1 of Vaughan [18], whenever $(\alpha,\beta)\in\mathfrak{W}(q,c,d)\subseteq\mathfrak{W}$, one has

$$f(\Lambda_j)-q^{-1}S_j(q,c,d)w_j(\alpha-c/q,\beta-d/q)\ll X^{1/2+\varepsilon}\quad(1\leqslant j\leqslant s). \tag{3.5}$$

On recalling the definition (3.3), and writing

$$W(\xi,\zeta)=\prod_{j=1}^s w(\lambda_j)\quad\text{and}\quad U(q,c,d)=q^{-s}\prod_{j=1}^s S_j(q,c,d),$$

we deduce from (3.5) that the estimate

$$H_s(\alpha,\beta)-U(q,c,d)W(\alpha-c/q,\beta-d/q)\ll P^{s-1}X^{1/2+\varepsilon} \tag{3.6}$$

holds uniformly in $(\alpha,\beta)\in\mathfrak{W}(q,c,d)\subseteq\mathfrak{W}$.

We next introduce truncated versions of the singular integral and singular series, which we define respectively by

$$\mathcal{J}(Y)=\iint_{\mathfrak{B}(Y)} W(\xi,\zeta)\,d\xi\,d\zeta\quad\text{and}\quad\mathfrak{S}(Y)=\sum_{1\leqslant q\leqslant Y}A(q), \tag{3.7}$$

in which we have written $\mathfrak{B}(Y)$ for the box $[-YP^{-3},YP^{-3}]^2$, and where

$$A(q)=\sum_{\substack{c=1\\(c,d,q)=1}}^q\sum_{d=1}^q U(q,c,d).$$

The measure of the major arcs $\mathfrak{W}(X)$ is $O(X^5P^{-6})$, so on recalling that $X = (\log P)^6$ and integrating over $\mathfrak{W}$, we infer from (3.6) that

$$\iint_{\mathfrak{W}} H_s(\alpha,\beta)\,d\alpha\,d\beta - \mathfrak{S}(X)\mathcal{J}(X) \ll P^{s-13/2}. \tag{3.8}$$

We discuss the truncated versions of the singular series and singular integral defined in (3.7) by making use of the analysis underlying Lemmata 12 and 13 of [8]. We note here that in the formulation of the latter, the exponential sums corresponding to the forms $\Lambda_2,\ldots,\Lambda_s$ are smooth Weyl sums rather than the present classical Weyl sums. This deviation from [8] demands at most cosmetic alterations to the argument of §7 of that paper, and we spare the reader the details. An examination of the proofs of these lemmata will confirm that it is sufficient that the maximum multiplicity of any of $\Lambda_1,\Lambda_2,\ldots,\Lambda_{13}$ is at most six amongst the latter forms. Since we assume that $r_1 \leqslant 6$, such is already guaranteed by the discussion in the preamble to the statement of our lemma. Putting

$$\mathcal{J} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(\xi,\zeta)\,d\xi\,d\zeta,$$

we find that the argument of the proof of Lemma 13 of [8] yields the estimate

$$\mathcal{J} - \mathcal{J}(X) \ll P^{s-6}X^{-1}.$$

Moreover, one finds that $\mathcal{J} = v_\infty P^{s-6}$, with v_∞ defined as in the discussion surrounding (1.2) above. Meanwhile, with v_p defined for each prime number p via the relation (1.3), it transpires that Lemma 12 of [8] supplies the formula

$$\mathfrak{S}(X) - \prod_p v_p \ll X^{-1/4}.$$

The asymptotic formula claimed in the statement of the lemma therefore follows at once from (3.8). That $\mathfrak{C}(\mathbf{a},\mathbf{b}) \gg 1$ whenever the system (1.1) possesses a non-trivial 7-adic solution is immediate from the aforementioned lemmata, on recalling our definition of the real s-tuple $\boldsymbol{\theta}$. This completes the proof of the lemma. □

4. The minor arc estimate

We next seek to establish the upper bound

$$\iint_{\mathfrak{w}} H_s(\alpha,\beta)\,d\alpha\,d\beta = o(P^{s-6}). \tag{4.1}$$

Making use of the conclusion of Lemma 3.1 and recalling that $[0,1)^2 \setminus \mathfrak{w} = \mathfrak{W}$, it follows from (4.1) that

$$\int_0^1\int_0^1 H_s(\alpha,\beta)\,d\alpha\,d\beta = \mathfrak{C}(\mathbf{a},\mathbf{b})P^{s-6} + o(P^{s-6}),$$

and the conclusion of Theorem 1.1 is delivered by (3.4). We begin by recalling an elementary observation from our earlier work, the proof of which is self-evident (see Lemma 5 of [8]).

Lemma 4.1. *Let k and N be natural numbers, and suppose that $\mathfrak{B} \subseteq \mathbb{R}^k$ is measurable. Let $\omega_i(\mathbf{z})$ $(0 \leqslant i \leqslant N)$ be complex-valued functions of $\mathfrak{B}$. Then whenever the functions $|\omega_0(\mathbf{z})\omega_j(\mathbf{z})^N|$ $(1 \leqslant j \leqslant N)$ are integrable on $\mathfrak{B}$, one has the upper bound*

$$\int_{\mathfrak{B}} |\omega_0(\mathbf{z})\omega_1(\mathbf{z})\dots\omega_N(\mathbf{z})|\, d\mathbf{z} \leqslant N \max_{1\leqslant j\leqslant N} \int_{\mathfrak{B}} |\omega_0(\mathbf{z})\omega_j(\mathbf{z})^N|\, d\mathbf{z}.$$

It is convenient in what follows to abbreviate, for each index m, the expression $|f(\Lambda_m)|$ simply to f_m, and likewise $|G_m(\alpha,\beta)|$ to G_m.

Lemma 4.2. *Suppose that the system (1.1) is of type* III *with $s \geqslant 14$ and $q_0 \geqslant 8$. Then in the setting described in §3, one has*

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\, d\alpha\, d\beta \ll P^{s-6}(\log P)^{\varepsilon-1}.$$

Proof. We begin with some simplifying analytic observations. Write $\mathcal{L} = \{\Lambda_1,\dots,\Lambda_{14}\}$, and recall that the number of equivalence classes in $\mathcal{L}$ is $t \geqslant 3$. By relabelling indices if necessary, we may suppose that representatives of these classes are $\widetilde{\Lambda}_i \in \mathcal{L}$ $(1 \leqslant i \leqslant t)$, and also that for each index i, the multiplicity of $\widetilde{\Lambda}_i$ amongst the elements of the set $\mathcal{L}$ is r_i. Then according to the discussion of the previous section, we may suppose that $\Lambda_1 \in [\widetilde{\Lambda}_1]$, and that

$$1 \leqslant r_t \leqslant r_{t-1} \leqslant \dots \leqslant r_1 \leqslant 6 \quad \text{and} \quad r_2 + r_3 + \dots + r_t = 14 - r_1. \tag{4.2}$$

Next, for a given index i with $1 \leqslant i \leqslant 14$, consider the linear forms Λ_{l_j} $(1 \leqslant j \leqslant r_i)$ equivalent to $\widetilde{\Lambda}_i$ from the set $\mathcal{L}$. Apply Lemma 4.1 with $N = r_i$, with f_{l_j} in place of ω_j $(1 \leqslant j \leqslant N)$, and with ω_0 replaced by the product of those f_m with $\Lambda_m \notin [\widetilde{\Lambda}_i]$ $(1 \leqslant m \leqslant 14)$, multiplied by $G_{14}(\alpha,\beta)$. We see that there is no loss of generality in supposing that $\Lambda_{l_j} = \widetilde{\Lambda}_i$ $(1 \leqslant j \leqslant r_i)$. By repeating this argument for successive equivalence classes, it follows that with a suitable choice of equivalence class representatives $\widetilde{\Lambda}_m$ $(1 \leqslant m \leqslant t)$, one has

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\, d\alpha\, d\beta \ll \iint_{\mathfrak{w}} G_{14}\widetilde{f}_1^{r_1}\widetilde{f}_2^{r_2}\dots\widetilde{f}_t^{r_t}\, d\alpha\, d\beta, \tag{4.3}$$

where we have abbreviated $|f(\widetilde{\Lambda}_m)|$ to $\widetilde{f}_m$ for each m.

We next apply a device employed in the proof of Lemma 6 of [8]. Let ν be a non-negative integer, and suppose that $r_{t-1} = r_t + \nu < 6$. Then we may apply Lemma 4.1 with $N = \nu + 2$, with $\widetilde{f}_{t-1}$ in place of ω_i $(1 \leqslant i \leqslant \nu+1)$ and $\widetilde{f}_t$ in place of ω_N, and with ω_0 set equal to

$$G_{14}\widetilde{f}_1^{r_1}\widetilde{f}_2^{r_2}\dots\widetilde{f}_{t-2}^{r_{t-2}}\widetilde{f}_{t-1}^{r_{t-1}-\nu-1}\widetilde{f}_t^{r_t-1}.$$

Here, and in what follows, we interpret the vanishing of any exponent as indicating that the associated exponential sum is deleted from the product. In this way, we obtain an upper bound of the shape (4.3), subject to the constraints (4.2), wherein either the parameter r_t is reduced, or else the parameter t is reduced. By repeating this process, therefore, we ultimately arrive at a situation in which $t = 3$ and $r_{t-1} = 6$, and then the constraints (4.2) imply that necessarily $(r_1, r_2, \dots, r_t) = (6, 6, 2)$.

On recalling (3.3) and (4.3), and making a trivial estimate for G_{14}, we may conclude at this point that

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\,d\alpha\,d\beta \ll P^{s-14} \iint_{\mathfrak{w}} \widetilde{f}_1^6 \widetilde{f}_2^6 \widetilde{f}_3^2\,d\alpha\,d\beta.$$

Making use of Theorem 2.5 in order to estimate the integral on the right hand side of this inequality, we obtain the estimate

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\,d\alpha\,d\beta \ll P^{s-14}(P^8(\log P)^{\varepsilon-1}),$$

and the conclusion of Lemma 4.2 follows at once. □

The proof of Theorem 1.1 for systems of type III is now immediate from the discussion in the preamble to the statement of Lemma 4.1. Having completed our discussion of systems of type III, we turn next to systems of type II. In this context, we remark that the discussion of §3 shows that a type II system with $q_0 \geqslant 8$ necessarily has at least 15 variables.

Lemma 4.3. *Suppose that the system (1.1) is of type* II *with* $s \geqslant 15$ *and* $q_0 \geqslant 8$. *Then in the setting described in* §3, *one has*

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\,d\alpha\,d\beta \ll P^{s-6}(\log P)^{\varepsilon-1/2}.$$

Proof. We begin by following the discussion initiating the proof of Lemma 4.2. In this way, we see that with a suitable choice of equivalence class representatives $\widetilde{\Lambda}_m$ $(1 \leqslant m \leqslant t)$, one has

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\,d\alpha\,d\beta \ll \iint_{\mathfrak{w}} G_{15} \widetilde{f}_1^{r_1} \widetilde{f}_2^{r_2} \widetilde{f}_3^{r_3}\,d\alpha\,d\beta.$$

Here, we have employed the same convention regarding the meaning of $\widetilde{f}_m$ as in the proof of Lemma 4.2. We note also that the discussion of type II systems in §3 reveals that $(r_1, r_2, r_3) = (7, 7, 1)$. An application of Schwarz's inequality, in combination with the trivial estimate $G_{15} = O(P^{s-15})$, therefore yields

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\,d\alpha\,d\beta \ll P^{s-15} \mathcal{K}_1^{1/2} \mathcal{K}_2^{1/2}, \tag{4.4}$$

where

$$\mathcal{K}_1 = \iint_{\mathfrak{w}} \widetilde{f}_1^6 \widetilde{f}_2^6 \widetilde{f}_3^2\,d\alpha\,d\beta \quad \text{and} \quad \mathcal{K}_2 = \int_0^1 \int_0^1 \widetilde{f}_1^8 \widetilde{f}_2^8\,d\alpha\,d\beta. \tag{4.5}$$

On recalling the hypothesis that $b_1 = a_2 = 0$, we see that for suitable non-zero integers a, b, c, d, one has

$$\widetilde{f}_1 = |f(a\alpha)|, \quad \widetilde{f}_2 = |f(b\beta)| \quad \text{and} \quad \widetilde{f}_3 = |f(c\alpha + d\beta)|.$$

On considering the underlying diophantine equations, it therefore follows from (4.5) together with Theorem 2 of Vaughan [16] that $\mathcal{K}_2 \ll (P^5)^2$. We now substitute this estimate together with the conclusion of Theorem 2.5 into (4.4) to obtain

$$\iint_{\mathfrak{w}} |H_s(\alpha,\beta)|\,d\alpha\,d\beta \ll P^{s-15}(P^8(\log P)^{\varepsilon-1})^{1/2}(P^{10})^{1/2},$$

and the conclusion of the lemma follows immediately. □

The proof of Theorem 1.1 for systems of type II now follows at once from the discussion in the preamble to the statement of Lemma 4.1. All that remains is to establish Theorem 1.1 for systems of type I. But given a system of type I with $q_0 \geqslant 8$, one has $T = 2$ and $R_1 \geqslant R_2 \geqslant 8$. In view of the hypothesis that $b_1 = a_2 = 0$, we may relabel variables in such a manner that for $1 \leqslant i \leqslant R_1$ one has $\Lambda_i = a_i\alpha$, and so that for $R_1 < j \leqslant R_1 + R_2$ one has $\Lambda_j = b_j\beta$. Classical methods (see Chapter 4 of Vaughan [18]) provide the asymptotic formula

$$\left(\int_{\mathfrak{M}_1} \prod_{i=1}^{R_1} f(a_i\alpha)\, d\alpha\right)\left(\int_{\mathfrak{M}_1} \prod_{j=R_1+1}^{R_1+R_2} f(b_j\beta)\, d\beta\right) = \mathfrak{C}(\mathbf{a},\mathbf{b})P^{s-6} + o(P^{s-6}).$$

In addition, when $(\mathfrak{U},\mathfrak{V})$ is one of $(\mathfrak{m}_1,\mathfrak{m}_1)$, $(\mathfrak{M}_1,\mathfrak{m}_1)$ and $(\mathfrak{m}_1,\mathfrak{M}_1)$, the methods of Vaughan [16] (see (2.9) above) supply the upper bound

$$\left(\int_{\mathfrak{U}} \prod_{i=1}^{R_1} f(a_i\alpha)\, d\alpha\right)\left(\int_{\mathfrak{V}} \prod_{j=R_1+1}^{R_1+R_2} f(b_j\beta)\, d\beta\right) \ll (P^{R_1-3})(P^{R_2-3})(\log P)^{\varepsilon-1}.$$

Thus we find that

$$\begin{aligned} N(P;\mathbf{a},\mathbf{b}) &= \left(\int_0^1 \prod_{i=1}^{R_1} f(a_i\alpha)\, d\alpha\right)\left(\int_0^1 \prod_{j=R_1+1}^{R_1+R_2} f(b_j\beta)\, d\beta\right) \\ &= \mathfrak{C}(\mathbf{a},\mathbf{b})P^{s-6} + o(P^{s-6}). \end{aligned}$$

This establishes Theorem 1.1 for type I systems, and completes the proof of the theorem.

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